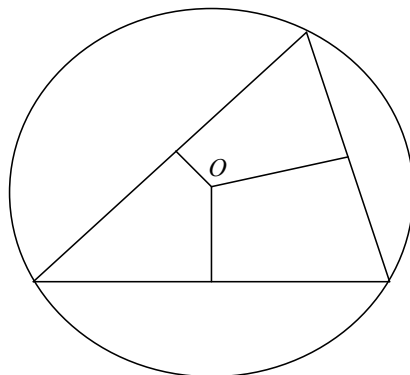


## The Euler Line of a Triangle

### The circumcentre of a triangle

The perpendicular bisectors of the sides of a triangle meet at a point  $O$ . This point is equidistant from the vertices of the triangle. Hence it is the centre of a circle which goes through the vertices of the triangle.

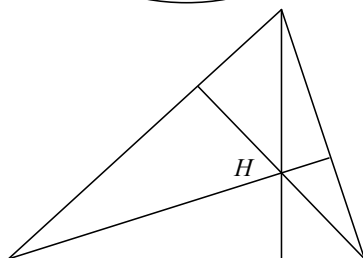
This circle is called the *circumcircle* of the triangle. The centre,  $O$ , of this circle is called the *circumcentre* of the triangle.



### The orthocentre of a triangle

The three altitudes of a triangle meet at a point,  $H$ . A proof of this will be found in the book mentioned below.

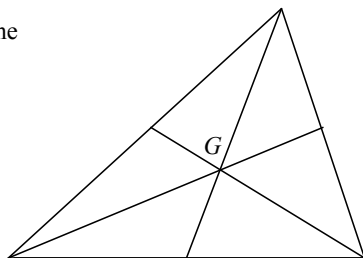
The point  $H$  is called the *orthocentre* of the triangle.



### The centroid of a triangle

The *medians* of a triangle are the lines joining the vertices to the midpoints of the opposite sides. The medians of a triangle meet at a point,  $G$ . A proof of this is also given in the book mentioned below. The point  $G$  is called the *centroid* of the triangle.

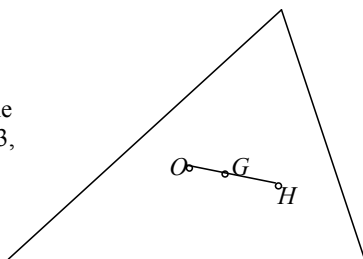
If a triangle is made of uniform material, its centre of gravity is at the centroid.  $G$  is also the centre of gravity of three equal point masses placed at the vertices of the triangle.



### The Euler Line

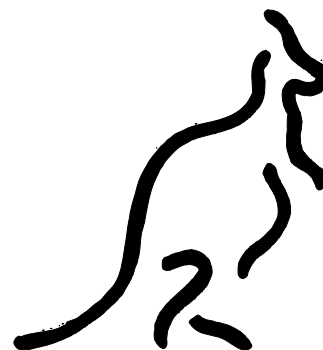
It is a remarkable fact that in any triangle the circumcentre, orthocentre and centroid lie on a line, called the *Euler Line* of the triangle, and the ratio  $OG:GH=1:2$  (Leonhard Euler, 1707-1783, was a very distinguished Swiss mathematician.)

You will find a proof of this in *Crossing the Bridge*, by Gerry Leversha, published by the UKMT in 2008. Go to <http://www.ukmt.org.uk> to order this and other UKMT publications.

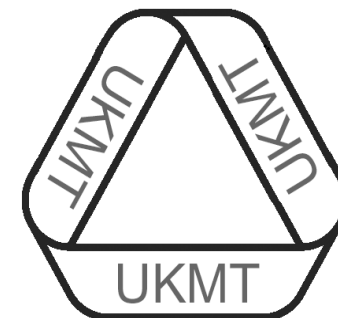


# Intermediate Mathematical Olympiad and Kangaroo (IMOK)

2009



*Kangourou des Mathématiques  
Kangaroo Papers*



*United Kingdom Mathematics Trust  
Olympiad Papers*

**United Kingdom Mathematics Trust**

The UKMT is a registered charity.

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# The European Kangaroo 2008

Questions from other countries

Once again, I am very grateful for the privilege of introducing this booklet, packed with mathematics. I also extend my heartiest congratulations! In order to be invited to take one of the follow-up events, you did very well indeed in the IMC in February and you should feel very pleased - no matter how challenging you actually found the paper you took recently. I hope this booklet is of use and interest to you, not only for you to revisit the paper that you took but also for you to have the chance of seeing all the other papers and, perhaps after attempting the questions, studying the 'official' solutions. There is much pleasure to be gained from taking every opportunity to widen your experience. In addition, I hope you find the biographical note about Arthur Cayley of interest too, although he lived and worked in a very different era to our own, he was certainly very gifted! In amongst all the work you do to master mathematical techniques, it is often overlooked that the mathematics you learn about was created or discovered by past mathematicians. In a similar way, the current generations of mathematicians are still discovering new mathematics - the search goes on, and on! Perhaps your future lies in that direction.

Mathematics is a magnificent subject. It is studied world wide and has been studied by humankind for thousands of years. I well remember attending a mathematics conference in Japan. We were given a Japanese text book, sadly I was able to make nothing at all of the Japanese writing but was delighted to see recognisable symbols throughout the book. Mathematics is hugely valuable, both for its own sake but also for the logic which underlies it and is embedded in the way a mathematician thinks. Such thinking skills are highly valued in our society but let us not lose sight of the beauty of mathematics - be that in some wonderful geometric diagram or in the way in which an algebraic proof is fitted together.

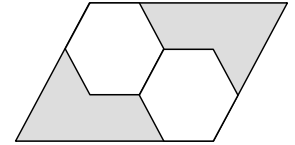
Finally, I should also mention that the UKMT competitions depend upon a great deal of voluntary effort by those setting and marking the papers plus some dedicated work by the staff in the UKMT office in Leeds. On your behalf, I think they should be thanked, and I do so publicly here. In addition, my thanks to you and also to the schools and homes which encourage you to take part. Good luck to you in the future.

Bill Richardson  
Chair of the UKMT Challenges Subtrust

## Italy

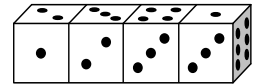
Il due esagoni che appaiono in figura sono regolari e l'area del parallelogramma vale 1. Quanto vale l'area della parte ombreggiata?

- A  $\frac{1}{2}$       B  $\frac{17}{36}$       C  $\frac{3}{7}$       D  $\frac{7}{12}$       E  $\frac{12}{23}$



## Slovakia

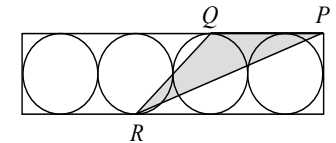
Na obrázku je vytvorený „had“ zo štyroch rovnakých kociek. Kocky sice majú na stenách 1, 2, 3, 4, 5 a 6 bodiek, ale súčet bodiek na protiľahlých stenách nemusí byť sedem. Aký je súčet bodiek na šiestich stenách, ktorými sa kocky navzájom dotýkajú?



- A 20      B 21      C 22      D 23      E 24

## Belarus

В прямоугольник вписаны 4 одинаковые окружности радиусом 6 см так, как показано на рис. (соседние окружности касаются друг друга). Найдите площадь треугольника  $PQR$ , где  $R$  и  $Q$  – указанные на рис. Точки касания двух из окружностей со сторонами прямоугольника, а  $P$  – одна из его вершин.



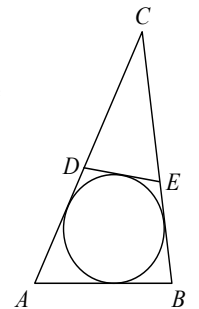
- A  $27 \text{ cm}^2$       B  $45 \text{ cm}^2$       B  $54 \text{ cm}^2$       Г  $108 \text{ cm}^2$       Д  $180 \text{ cm}^2$

## Austria

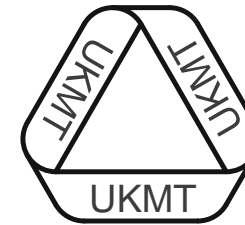
In das Dreieck  $ABC$  mit  $\overline{AB} = 3$ ,  $\overline{BC} = 5$  und  $\overline{CA} = 6$  sei ein Kreis eingeschrieben (s. Abb., Zeichnung ist nicht maßstabgerecht). Ferner sei die Strecke  $\overline{DE}$  Tangente an den Kreis.

Dann ist der Umfang des Dreiecks  $CDE$  gleich

- A 4      B 6      C 7      D 8      E 9



Answers: A A Г D



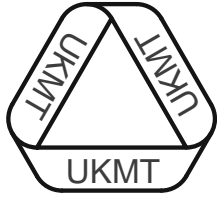
This booklet contains the questions and solutions for the follow-up competitions of the UKMT Intermediate Mathematical Challenge held in February 2009.

For the age ranges covered, see the details on the relevant paper.

As is usual, it is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

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**EUROPEAN 'KANGAROO' MATHEMATICAL CHALLENGE  
'GREY'**

**Thursday 19th March 2009**

**Organised by the United Kingdom Mathematics Trust and the  
Association Kangourou Sans Frontières**

*This competition is being taken by 5 million students in over 40 countries worldwide.*

RULES AND GUIDELINES (to be read before starting):

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No answers, or personal details, may be entered after the allowed hour is over.
3. The use of rough paper is allowed; **calculators** and measuring instruments are **forbidden**.
4. Candidates in England and Wales must be in School Year 9 or below.  
Candidates in Scotland must be in S2 or below.  
Candidates in Northern Ireland must be in School Year 10 or below.
5. **Use B or HB pencil only**. For each question mark *at most one* of the options A, B, C, D, E on the Answer Sheet. Do not mark more than one option.
6. Five marks will be awarded for each correct answer to Questions 1 - 15.  
Six marks will be awarded for each correct answer to Questions 16 - 25.
7. *Do not expect to finish the whole paper in 1 hour*. Concentrate first on Questions 1-15. When you have checked your answers to these, have a go at some of the later questions.
8. The questions on this paper challenge you **to think**, not to guess. Though you will not lose marks for getting answers wrong, you will undoubtedly get more marks, and more satisfaction, by doing a few questions carefully than by guessing lots of answers.

*Enquiries about the European Kangaroo should be sent to: Maths Challenges Office,  
School of Mathematics, University of Leeds, Leeds, LS2 9JT.  
(Tel. 0113 343 2339)  
<http://www.ukmt.org.uk>*

1. Which of the following calculations results in an even number?

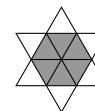
A  $2 \times 0 + 0 + 9$    B  $2 + 0 + 0 + 9$    C  $200 - 9$    D  $200 + 9$    E  $200 \times 9$

2. The star on the right is formed from 12 identical equilateral triangles.

The length of the perimeter of the star is 36 cm.

What is the length of the perimeter of the shaded hexagon?

A 12 cm   B 18 cm   C 24 cm   D 30 cm   E 36 cm



3. The product of four different positive integers is 100. What is the sum of these four integers?

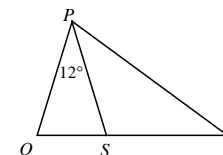
A 10                      B 12                      C 14                      D 15                      E 18

4. In the diagram on the right,  $QSR$  is a straight line,

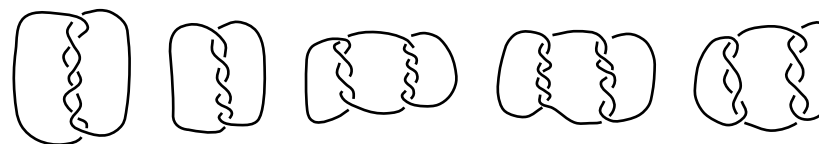
$\angle QPS = 12^\circ$  and  $PQ = PS = RS$ .

What is the size of  $\angle QPR$ ?

A  $36^\circ$    B  $42^\circ$    C  $54^\circ$    D  $60^\circ$    E  $84^\circ$



5. Which of the following knots consist of more than one loop of rope?



*P*                      *Q*                      *R*                      *S*                      *T*

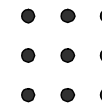
A *P*, *R* and *T*                      B *R*, *S* and *T*                      C *P*, *R*, *S* and *T*

D all of *P*, *Q*, *R*, *S* and *T*                      E none of *A*, *B*, *C* or *D*

6. How many positive integers  $n$  exist for which  $n^2$  has the same number of digits as  $n^3$ ?

A 0                      B 3                      C 4                      D 9                      E infinitely many

7. The diagram on the right shows nine points in a square array. What is the smallest number of points that need to be removed in order that no three of the remaining points are in a straight line?



A 1                      B 2                      C 3                      D 4                      E 7

8. Nick measured all six of the angles in two triangles – one acute-angled and one obtuse-angled.

He remembered that four of the angles were  $120^\circ$ ,  $80^\circ$ ,  $55^\circ$  and  $10^\circ$ .

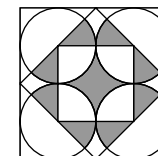
What is the size of the smallest angle of the acute-angled triangle?

A  $5^\circ$                       B  $10^\circ$                       C  $45^\circ$                       D  $55^\circ$                       E more information needed

9. The diagram shows four circles each of which touches the largest square and two adjacent circles. A second square has its vertices at the midpoints of the sides of the largest square and the central square has its vertices at the centres of the circles.

What is the ratio of the total shaded area to the area of the outer square?

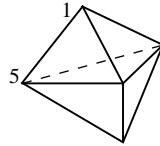
A  $\pi : 12$                       B  $1 : 4$                       C  $(\pi + 2) : 16$                       D  $1 : 3$                       E  $\pi : 4$



10. A magical island is inhabited entirely by knights (who always tell the truth) and knaves (who always tell lies). One day 25 of the islanders were standing in a queue. The first person in the queue said that everybody behind was a knave. Each of the others in the queue said that the person immediately in front of them in the queue was a knave. How many knights were there in the queue?

A 0      B 12      C 13      D 24      E more information needed

11. The diagram shows a solid with six triangular faces. At each vertex there is a number and two of the numbers are 1 and 5, as shown. For each face the sum of the numbers at the three vertices of each face is calculated, and all the sums are the same. What is the sum of all five numbers?



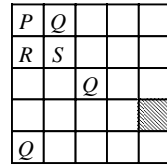
A 9      B 12      C 17      D 18      E 24

12. In the equation  $\frac{E \times I \times G \times H \times T}{F \times O \times U \times R} = T \times W \times O$ , the same letter stands for the same digit and different letters stand for different digits.

How many different possible values are there for the product  $T \times H \times R \times E \times E$ ?

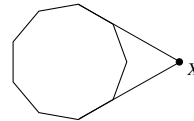
A 1      B 2      C 3      D 4      E 5

13. In each of the squares in the grid, one of the letters P, Q, R and S must be entered in such a way that adjacent squares (whether connected by an edge or just a corner) do not contain the same letter. Some of the letters have already been entered as shown. What are the possibilities for the letter in the shaded square?



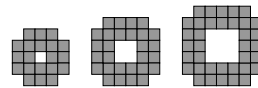
A only Q      B only R      C only S  
D either R or S, but no others      E it is impossible to complete the grid

14. The diagram shows a regular 9-sided polygon (a nonagon or an enneagon) with two of the sides extended to meet at the point X. What is the size of the acute angle at X?



A  $40^\circ$       B  $45^\circ$       C  $50^\circ$       D  $55^\circ$       E  $60^\circ$

15. The diagram shows the first three patterns in a sequence in which each pattern has a square hole in the middle. How many small shaded squares are needed to build the **tenth** pattern in the sequence?

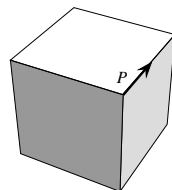


A 76      B 80      C 84      D 92      E 100

16. How many ten-digit numbers are there which contain only the digits 1, 2 or 3, and in which any pair of adjacent digits differs by 1?

A 16      B 32      C 64      D 80      E 100

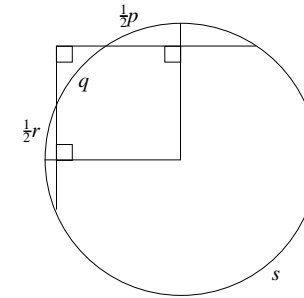
17. An ant crawls carefully around the edges of a cube, starting at point P and in the direction of the arrow. At the end of the first edge he chooses to go either left or right. He then turns the other way at the end of the next edge and continues like this, turning right or left alternately at the end of each successive edge. After how many edges does the ant return to point P for the first time?



A 2      B 6      C 8      D 9      E 12

### Second solution

We construct the two radii perpendicular to the two given chords, as shown in the diagram.



Since a radius perpendicular to a chord bisects the chord, it also bisects the corresponding arc. So we have arcs of length  $\frac{1}{2}p$  and  $\frac{1}{2}r$ , as shown.

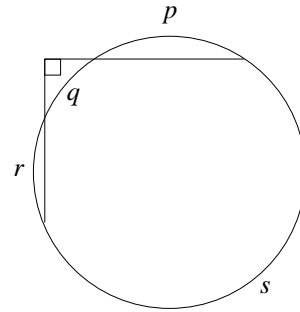
Now the diagram contains a quadrilateral with three right angles, so the fourth angle is also a right angle and therefore the two radii are perpendicular. Hence the radii determine a quarter of the circle, and so

$$p + q + r + s = 4 \times \left(\frac{1}{2}p + q + \frac{1}{2}r\right),$$

which gives

$$s = p + 3q + r.$$

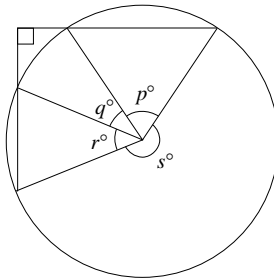
6. In the figure,  $p$ ,  $q$ ,  $r$  and  $s$  are the lengths of four arcs which together form the circumference of the circle. Find, in simplified form, an expression for  $s$  in terms of  $p$ ,  $q$  and  $r$ .



*First solution*

First note that the length of an arc of a circle is proportional to the angle subtended at the centre. Hence, by appropriate choice of radius, we can arrange that the arc lengths are actually equal to these angles. (If this is not done, the calculation proceeds in the same way, but there is some cancelling to do at the end.)

The diagram shows these four angles at the centre of the circle.



Now there are two relationships between  $p$ ,  $q$ ,  $r$  and  $s$ . First there is the obvious one

$$p + q + r + s = 360. \quad (1)$$

We also have the angle sum of a quadrilateral; one of the angles is  $90^\circ$ , one is  $(p + q + r)^\circ$ , and the others can be calculated in terms of  $p$  and  $r$  using the fact that there are two isosceles triangles. This gives

$$(p + q + r) + (90 - \frac{1}{2}p) + 90 + (90 - \frac{1}{2}r) = 360$$

and so

$$p + 2q + r = 180.$$

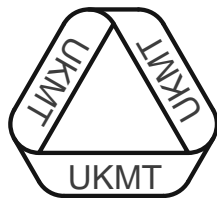
Hence, from (1),

$$p + q + r + s = 2p + 4q + 2r$$

and so

$$s = p + 3q + r.$$

18. The fractions  $\frac{1}{3}$  and  $\frac{1}{5}$  have been placed on the number-line shown on the right. At which position should the fraction  $\frac{1}{4}$  be placed?
- 
- A  $a$     B  $b$     C  $c$     D  $d$     E  $e$
19. Three cuts are made through a large cube to make eight smaller cuboids, as shown in the diagram on the right. What is the ratio of the total surface area of these eight cuboids to the total surface area of the original cube?
- 
- A 1 : 1    B 4 : 3    C 3 : 2    D 2 : 1    E 4 : 1
20. When Tina chose a number  $N$  and wrote down all of its factors, apart from 1 and  $N$ , she noticed that the largest of the factors in the list was 45 times the smallest factor in the list. How many numbers  $N$  could Tina have chosen for which this is the case?
- A 0    B 1    C 2    D more than 2    E more information needed
21. A square has been dissected into 2009 smaller squares so that the sides of each smaller square are a whole number of units long. What is the shortest possible length of the side of the original square?
- A 44    B 45    C 46    D 47    E 48
22. In a quadrilateral  $PQRS$ ,  $PQ = 2006$ ,  $QR = 2008$ ,  $RS = 2007$  and  $SP = 2009$ . Which interior angles of the quadrilateral are necessarily less than  $180^\circ$ ?
- A  $P$ ,  $Q$ ,  $R$  and  $S$     B  $P$ ,  $Q$  and  $R$     C  $Q$ ,  $R$  and  $S$     D  $P$ ,  $Q$  and  $S$     E  $P$ ,  $R$  and  $S$
23. If I place a 6 cm by 6 cm square on a triangle, I can cover up to 60% of the triangle. If I place the triangle on the square, I can cover up to  $\frac{2}{3}$  of the square. What is the area of the triangle?
- A 22.8 cm<sup>2</sup>    B 24 cm<sup>2</sup>    C 36 cm<sup>2</sup>    D 40 cm<sup>2</sup>    E 60 cm<sup>2</sup>
24. Peter wishes to write down a list of different positive integers less than or equal to 10 in such a way that for each pair of adjacent numbers one of the numbers is divisible by the other. What is the length of the longest list that Peter could write down?
- A 6    B 7    C 8    D 9    E 10
25. In a triangle  $PQR$ ,  $\angle PQR = 20^\circ$  and  $\angle PRQ = 40^\circ$ . The point  $S$  lies on  $QR$  so that  $PS$  bisects  $\angle QPR$ , and  $PS = 2$  units. How many units longer than  $PQ$  is  $QR$ ?
- A 1    B 1.5    C 2    D 4    E more information needed



**EUROPEAN 'KANGAROO' MATHEMATICAL CHALLENGE  
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(Tel. 0113 343 2339)  
<http://www.ukmt.org.uk>*

Now  $\angle CXP = \angle DAP$ , by the external angle of a cyclic quadrilateral, and also  $\angle CXP = \angle XPA$ , by alternate angles. Hence  $\angle XPA = \angle DAP = \angle CBP$  and so  $XP$  is parallel to  $CB$  and  $PBCX$  is a parallelogram. Similarly,  $APYD$  is a parallelogram. In particular,  $CX = PB$  and  $DY = AP$ .

Now we use the tangent-secant theorem for each circle. This gives

$$\begin{aligned} CP^2 &= CX \times CD \\ &= PB \times CD \end{aligned}$$

and

$$\begin{aligned} DP^2 &= DC \times DY \\ &= DC \times AP. \end{aligned}$$

The last stage of the proof proceeds in the same way as the first solution.

*Remark*

When the three marked angles are right angles the result reduces to Pythagoras' theorem.

5. A lottery involves five balls being selected from a drum. Each ball has a different positive integer printed on it.  
Show that, whichever five balls are selected, it is always possible to choose three of them so that the sum of the numbers on these three balls is a multiple of 3.

*Solution*

It is worth observing that, for the purposes of this question, we can replace the numbers on the balls by their remainders when they are divided by 3. This is because, for any two numbers  $a$  and  $b$ , their sum  $a + b$  is a multiple of 3 if, and only if, the sum of their remainders is divisible by 3. To prove this, suppose that, when divided by 3,  $a$  has remainder  $r$  and  $b$  has remainder  $s$ . Then we can write  $a = 3m + r$  and  $b = 3n + s$ , for some integers  $m$  and  $n$ . Therefore

$$a + b = 3m + 3n + r + s = 3(m + n) + r + s \text{ and this is divisible by 3 if, and only if, } r + s \text{ is divisible by 3.}$$

Using this idea, any selection of five balls will result in a set of five numbers, each of which is 0 or 1 or 2.

If three of the balls have the same number, then their sum will be divisible by 3, and we are done.

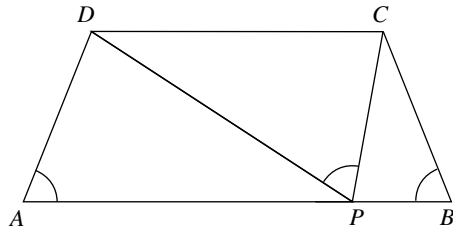
If not, then there is one pair of numbers equal to  $x$ , another pair equal to  $y$  and a fifth number equal to  $z$ , where  $x$ ,  $y$  and  $z$  are 0, 1 and 2, in some order. This means that there are three balls numbered 0, 1 and 2, and the sum of these numbers is divisible by 3.

Hence, in either case, we have a choice of three of the five balls whose sum is a multiple of 3.

4. In a trapezium  $ABCD$  the sides  $AB$  and  $DC$  are parallel and  $\angle BAD = \angle ABC < 90^\circ$ . Point  $P$  lies on  $AB$  with  $\angle CPD = \angle BAD$ . Prove that  $PC^2 + PD^2 = AB \times DC$ .

*Solutions*

We begin with a diagram showing the information given in the question. The three marked angles are equal.



*First solution*

Since  $AB$  is parallel to  $DC$ , the alternate angles  $\angle DCP$  and  $\angle CPB$  are equal. Hence the two triangles  $\triangle CDP$  and  $\triangle PCB$  are similar.

Exactly the same argument shows that  $\triangle DPA$  is also similar to  $\triangle CDP$ . We therefore have three similar triangles. Notice that we have been careful to describe the triangles in the correct order of vertices, so that the ratios of sides can now be read off conveniently.

Now corresponding sides are in the same ratio, so we have

$$\frac{PB}{PC} = \frac{PC}{DC} \quad \text{and} \quad \frac{AP}{PD} = \frac{PD}{DC},$$

and it follows that

$$PC^2 = PB \times DC \quad \text{and} \quad PD^2 = AP \times DC.$$

Adding these two equations, we obtain

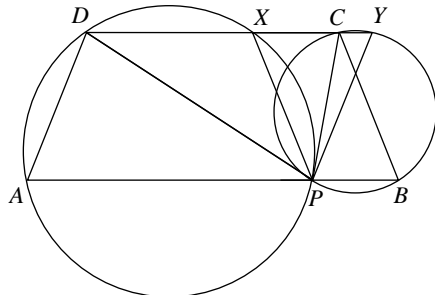
$$\begin{aligned} PC^2 + PD^2 &= (AP + PB) \times DC \\ &= AB \times DC. \end{aligned}$$

*Second solution*

This proof starts by using the converse of the alternate segment theorem. Because  $\angle DAB = \angle CPD$  it follows that  $CP$  is a tangent at  $P$  to the circumcircle of  $\triangle DAP$ .

Similarly,  $DP$  is a tangent at  $P$  to the circumcircle of  $\triangle BCP$ .

Let  $DC$  (extended) meet the two circles at  $X$  and  $Y$ , as shown.



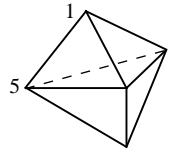
1. The Woomera Marathon had 2009 participants. The number of participants beaten by Kanga was three times the number that beat Kanga. In what position did Kanga finish the marathon?

A 500th    B 501st    C 503rd    D 1503rd    E 1507th

2. What is the value of  $\frac{1}{2}$  of  $\frac{2}{3}$  of  $\frac{3}{4}$  of  $\frac{4}{5}$  of  $\frac{5}{6}$  of  $\frac{6}{7}$  of  $\frac{7}{8}$  of  $\frac{8}{9}$  of  $\frac{9}{10}$  of 1000?

A 100    B 200    C 250    D 300    E none of these

3. The diagram shows a solid with six triangular faces. At each vertex there is a number. Two of the numbers are 1 and 5 as shown. For each face the sum of the numbers at the three vertices of that face is calculated, and all the sums are found to be the same. What is the sum of all five numbers at the vertices?

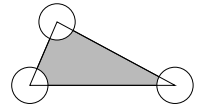


A 9    B 12    C 17    D 18    E 24

4. How many positive integers  $n$  exist for which  $n^2$  has the same number of digits as  $n^3$ ?

A 0    B 3    C 4    D 9    E infinitely many

5. The diagram shows a triangle and three circles whose centres are at the vertices of the triangle. The area of the triangle is  $80 \text{ cm}^2$  and each of the circles has radius 2 cm. What is the area, in  $\text{cm}^2$ , of the shaded area?



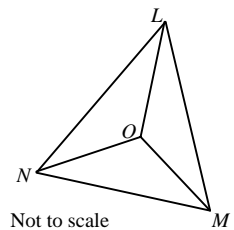
A 76    B  $80 - 2\pi$     C  $40 - 4\pi$     D  $80 - \pi$     E 78 $\pi$

6. Leonard writes down a sequence of numbers. After the first two numbers, each number is the sum of the previous two numbers in the sequence. The fourth number is six and the sixth number is fifteen. What is the seventh number in the sequence?

A 9    B 16    C 21    D 22    E 24

7. The three angle bisectors of triangle  $LMN$  meet at a point  $O$  as shown. Angle  $LMN$  is  $68^\circ$ . What is the size of angle  $LOM$ ?

A  $120^\circ$     B  $124^\circ$     C  $128^\circ$     D  $132^\circ$     E  $136^\circ$

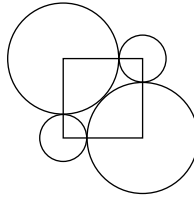


Not to scale

8. Mary sits four tests, each of which is out of 5 marks. Mary's average over the four tests is 4 marks. Which one of the following statements cannot be true?

A Mary obtained a mark of 4 out of 5 in each test  
 B Mary obtained a mark of 4 out of 5 twice  
 C Mary obtained a mark of 1 out of 5 once  
 D Mary obtained a mark of 3 out of 5 twice  
 E Mary obtained a mark of 3 out of 5 three times

9. A magical island is inhabited entirely by knights (who always tell the truth) and knaves (who always tell lies). One day 25 of the islanders were standing in a queue. The first person in the queue said that everybody behind was a knave. Each of the others in the queue said that the person immediately in front of them in the queue was a knave. How many knights were there in the queue?
- A 0      B 12      C 13      D 24      E more information needed
10. We define  $a \oplus b = ab + a + b$ . Given that  $3 \oplus 5 = 2 \oplus x$ , what is the value of  $x$ ?
- A 3      B 4      C 5      D 6      E 7
11. The diagram shows two identical large circles and two identical smaller circles whose centres are at the corners of a square. The two large circles are touching, and they each touch the two smaller circles. The radius of the small circles is 1 cm. What is the radius of a large circle in centimetres?
- A  $1 + \sqrt{2}$     B  $\sqrt{5}$       C  $\sqrt{2}$       D  $\frac{5}{2}$       E  $\frac{4}{5}\pi$
12. How many integers  $n$  exist such that the difference between  $\sqrt{n}$  and 10 is less than 1?
- A 19      B 20      C 39      D 40      E 41
13. Peter wishes to write down a list of different positive integers less than or equal to 10 in such a way that for each pair of adjacent numbers one of the numbers is divisible by the other. What is the length of the longest list that Peter could write down?
- A 6      B 7      C 8      D 9      E 10
14. Three circular hoops are joined together so that they intersect at right-angles as shown. A ladybird lands on an intersection and crawls around the outside of the hoops by repeating this procedure: she travels along a quarter-circle, turns  $90^\circ$  to the right, travels along a quarter-circle and turns  $90^\circ$  to the left. Proceeding in this way, how many quarter-circles will she travel along before she first returns to her starting point?
- A 6      B 9      C 12      D 15      E 18
15. Which of these decimals is less than  $\frac{2009}{2008}$  but greater than  $\frac{20009}{20008}$ ?
- A 1.01      B 1.001      C 1.0001      D 1.00001      E 1.000001
16. If  $a = 2^{25}$ ,  $b = 8^8$  and  $c = 3^{11}$ , then which of these statements is true?
- A  $a < b < c$     B  $b < a < c$     C  $b < c < a$     D  $c < a < b$     E  $c < b < a$
17. How many ten-digit numbers are there which contain only the digits 1, 2 or 3, and in which any pair of adjacent digits differs by 1?
- A 16      B 32      C 64      D 80      E 100



and therefore

$$5xy - 8xz = 3yz. \quad (5)$$

But, from (3),

$$3yz = 2y + 2z. \quad (6)$$

Putting (4), (5) and (6) together, we have

$$6y - 6z = 2y + 2z$$

and so

$$y = 2z.$$

Now, substituting  $y = 2z$  in (3) and cancelling  $z$ , we have

$$\frac{6z}{3} = 2,$$

so  $z = 1$  and  $y = 2$ . Then  $x = 3$ , by using (5), for example.

Note that it is now necessary to check that this triple of values works for all the equations. All we have shown so far is that, if the equations are true, then the only possible values of  $x$ ,  $y$  and  $z$  are 3, 2 and 1; this does not mean that, if  $x$ ,  $y$  and  $z$  are 3, 2 and 1, then the equations are true—for example, the equations may have *no* solutions.

*Second solution*

Since none of  $x$ ,  $y$  or  $z$  is zero, we can define

$$a = \frac{1}{x}, \quad b = \frac{1}{y} \quad \text{and} \quad c = \frac{1}{z}.$$

By taking the reciprocals of each side of the three equations, we obtain

$$b + a = \frac{5}{6}$$

$$a + c = \frac{4}{3}$$

$$c + b = \frac{3}{2}.$$

Hence, by adding, we deduce that

$$a + b + c = \frac{11}{6},$$

so  $c = 1$ ,  $b = \frac{1}{2}$  and  $a = \frac{1}{3}$ . Therefore  $x = 3$ ,  $y = 2$  and  $z = 1$ .

Again we should check these values satisfy the original equations, or show that our logic is reversible.

used, it must be made very clear that the check is done properly in every case. For example, one might exclude 5929 by showing that it is a multiple of 7 and thus has remainder 0. It would not be enough simply to claim that this check had been made without giving details in each case.

The task can be reduced by, for instance, excluding the even squares, leaving only 34 numbers to check, or realising that the number has to end in a 1 to satisfy the conditions for divisibility by 2 and 5. However, by the time you are applying this logic, you might as well go the whole way and use the method given above.

3. Solve the simultaneous equations

$$\frac{5xy}{x+y} = 6$$

$$\frac{4xz}{x+z} = 3$$

$$\frac{3yz}{y+z} = 2.$$

#### Solutions

It is worth beginning with the observation that none of  $x$ ,  $y$  or  $z$  can be zero, since that would immediately invalidate the equations. This allows us, in subsequent work, to cancel a factor of  $x$ ,  $y$  or  $z$  from both sides of an equation. Note also that we should not assume that  $x$ ,  $y$  or  $z$  are whole numbers.

#### First solution

Label the equations as follows:

$$\frac{5xy}{x+y} = 6 \quad (1)$$

$$\frac{4xz}{x+z} = 3 \quad (2)$$

$$\frac{3yz}{y+z} = 2. \quad (3)$$

From (1),

$$5xy = 6x + 6y$$

and from (2),

$$8xz = 6x + 6z,$$

so

$$5xy - 8xz = 6y - 6z. \quad (4)$$

We can, however, also deduce from (1) and (2) that

$$\frac{5xy}{x+y} = \frac{8xz}{x+z},$$

so, cancelling  $x$ ,

$$5y(x+z) = 8z(x+y)$$

18. Roo has glued 2009 unit cubes together to form a cuboid. He opens a pack containing 2009 stickers and he has enough to place one sticker on each exposed face of each unit cube. How many stickers does he have left?

A fewer than 49    B 49    C 763    D 1246    E more than 1246

19. When Tina chose a number  $N$  and wrote down all of its factors, apart from 1 and  $N$ , she noticed that the largest of the factors in the list was 45 times the smallest factor in the list.

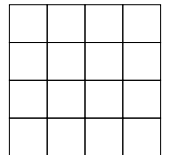
How many numbers  $N$  could Tina have chosen for which this is the case?

A 0    B 1    C 2    D more than 2    E more information needed

20. A grocer places some oranges, peaches, apples and bananas in a row so that, somewhere in the row, each type of fruit can be found next to each other type of fruit. What is the smallest possible number of fruits in the row?

A 7    B 8    C 12    D 16    E 32

21. Barbara wants to place draughts on a  $4 \times 4$  board in such a way that the number of draughts in each row and in each column are all different (she may place more than one draught in a square, and a square may be empty). What is the smallest number of draughts that she would need?



A 14    B 16    C 21    D 28    E 32

22. What is the smallest integer  $n$  such that the product

$$(2^2 - 1)(3^2 - 1)(4^2 - 1) \dots (n^2 - 1)$$

is a perfect square?

A 6    B 8    C 16    D 27    E none of these

23. A kangaroo is sitting in the Australian outback. He plays a game in which he may only jump 1 metre at a time, either North, East, South or West. At how many different points could he end up after 10 jumps?

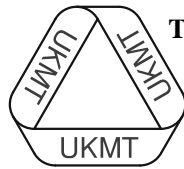
A 100    B 121    C 400    D 441    E none of these

24. Shakil wants to remove numbers from the set  $\{1, 2, 3, \dots, 16\}$  so that no two remaining numbers add to make a perfect square. What is the smallest number of numbers that he needs to remove?

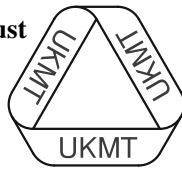
A 6    B 7    C 8    D 9    E 10

25. A prime number is called 'strange' if either it is a one-digit prime, or if each of the numbers obtained by removing its first digit or its last digit are themselves strange primes. How many strange primes are there?

A 6    B 7    C 8    D 9    E 11



The United Kingdom Mathematics Trust



## Intermediate Mathematical Olympiad and Kangaroo (IMOK)

### Olympiad Cayley/Hamilton/Maclaurin Papers

Thursday 19th March 2009

#### READ THESE INSTRUCTIONS CAREFULLY BEFORE STARTING

1. Time allowed: 2 hours.
2. **The use of calculators, protractors and squared paper is forbidden.** Rulers and compasses may be used.
3. Solutions must be written neatly on A4 paper. Sheets must be STAPLED together in the top left corner with the Cover Sheet on top.
4. Start each question on a fresh A4 sheet.  
You may wish to work in rough first, then set out your final solution with clear explanations and proofs. **Do not hand in rough work.**
5. Answers must be FULLY SIMPLIFIED, and EXACT. They may contain symbols such as  $\pi$ , fractions, or square roots, if appropriate, but NOT decimal approximations.
6. Give full written solutions, including mathematical reasons as to why your method is correct.  
Just stating an answer, even a correct one, will earn you very few marks; also, incomplete or poorly presented solutions will not receive full marks.
7. **These problems are meant to be challenging!** The earlier questions tend to be easier; the last two questions are the most demanding.  
Do not hurry, but spend time working carefully on one question before attempting another. Try to finish whole questions even if you cannot do many: you will have done well if you hand in full solutions to two or more questions.

#### DO NOT OPEN THE PAPER UNTIL INSTRUCTED BY THE INVIGILATOR TO DO SO!

The United Kingdom Mathematics Trust is a Registered Charity.

*Enquiries should be sent to: Maths Challenges Office,*

*School of Mathematics, University of Leeds, Leeds, LS2 9JT.*

*(Tel. 0113 343 2339)*

*<http://www.ukmt.org.uk>*

## Solutions to the Olympiad Maclaurin Paper

1. Five numbers are arranged in increasing order. As they get larger the difference between adjacent numbers doubles.  
The average of the five numbers is 11 more than the middle number. The sum of the second and fourth numbers is equal to the largest number.  
What is the largest number?

#### Solution

Let  $a$  be the smallest number and  $d$  be the difference between the first and second numbers. Then the other differences are  $2d$ ,  $4d$  and  $8d$ , so the five numbers are  $a$ ,  $a + d$ ,  $a + 3d$ ,  $a + 7d$  and  $a + 15d$ .

The condition that 'the average of the five numbers is 11 more than the middle number' gives

$$\frac{5a + 26d}{5} = a + 3d + 11,$$

which means that  $d = 5$ .

The condition that 'the sum of the second and fourth numbers is equal to the largest number' now gives

$$2a + 8d = a + 15d$$

and hence  $a = 35$ .

So the five numbers are 35, 40, 50, 70 and 110. Hence the largest number is 110.

#### Remark

Slightly neater algebra is obtained if instead we define  $a$  and  $d$  so that the numbers are represented as  $a - 3d$ ,  $a - 2d$ ,  $a$ ,  $a + 4d$  and  $a + 12d$ .

2. Miko always remembers his four-digit PIN (personal identification number) because
  - (a) it is a perfect square, and
  - (b) it has the property that, when it is divided by 2, or 3, or 4, or 5, or 6, or 7, or 8, or 9, there is always a remainder of 1.
 What is Miko's PIN?

#### Solution

Since the required number has four digits, it is between 1000 and 9999.

Let the number be  $N$ . We know that  $N$  is a perfect square, and also that  $N - 1$  is divisible by 2, 3, 4, 5, 6, 7, 8 and 9. Hence  $N - 1$  is divisible by the lowest common multiple of these numbers, which is  $8 \times 9 \times 5 \times 7 = 2520$ .

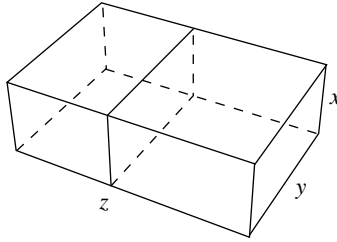
There are therefore only three possible values for  $N$  in the range, namely 2521, 5041 and 7561, and of these only the middle one is a perfect square, being  $71^2$ . Hence Miko's PIN is 5041.

#### Remark

It would be possible to start by listing all the four-digit perfect squares, and then checking them one by one for the condition (b) concerning remainders. This is, however, not a very efficient method, since there are 68 numbers to check. If a method like this is

*Second solution*

Let the large cuboid have dimensions  $x$ ,  $y$  and  $z$ , as shown.



Now the total surface area  $T$  of the two original cuboids is equal to the surface area of the large cuboid added to the area of the two faces which are joined together. But the surface area of the large cuboid is  $\frac{2}{3}T$ , hence the area of the two faces which are joined together is  $\frac{1}{3}T$ , that is,  $\frac{1}{3}$  of the surface area of the large cuboid.

Therefore

$$2xy = \frac{1}{3}(2xy + 2yz + 2zx)$$

so that

$$6xy = 2xy + 2yz + 2zx$$

and hence

$$2xy = yz + zx.$$

Dividing by  $xyz$ , we obtain, as required,

$$\frac{2}{z} = \frac{1}{x} + \frac{1}{y}.$$

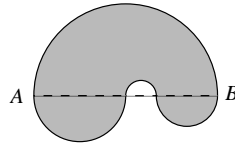
- *Do not hurry, but spend time working carefully on one question before attempting another.*
- *Try to finish whole questions even if you cannot do many.*
- *You will have done well if you hand in full solutions to two or more questions.*
- *Answers must be FULLY SIMPLIFIED, and EXACT. They may contain symbols such as  $\pi$ , fractions, or square roots, if appropriate, but NOT decimal approximations*
- *Give full written solutions, including mathematical reasons as to why your method is correct.*
- *Just stating an answer, even a correct one, will earn you very few marks.*
- *Incomplete or poorly presented solutions will not receive full marks.*
- ***Do not hand in rough work.***

## Olympiad Cayley Paper

All candidates must be in *School Year 9 or below (England and Wales), S2 or below (Scotland), or School Year 10 or below (Northern Ireland)*.

1. An aquarium contains 280 tropical fish of various kinds. If 60 more clownfish were added to the aquarium, the proportion of clownfish would be doubled. How many clownfish are in the aquarium?

2. The boundary of the shaded figure consists of four semicircular arcs whose radii are all different. The centre of each arc lies on the line  $AB$ , which is 10 cm long. What is the length of the perimeter of the figure?



3. Two different rectangles are placed together, edge-to-edge, to form a large rectangle. The length of the perimeter of the large rectangle is  $\frac{2}{3}$  of the total perimeter of the original two rectangles.

Prove that the final rectangle is in fact a square.

4. In the rectangle  $ABCD$ , the side  $AB$  has length  $\sqrt{2}$  and the side  $AD$  has length 1. Let the circle with centre  $B$  and passing through  $C$  meet  $AB$  at  $X$ .

Find  $\angle ADX$  (in degrees).

5. Two candles are the same height. The first takes 10 hours to burn completely whilst the second takes 8 hours to burn completely.

Both candles are lit at midday. At what time is the height of the first candle twice the height of the second candle?

6. Teams A, B, C and D competed against each other once. The results table was as follows:

Team	Win	Draw	Loss	Goals for	Goals against
A	3	0	0	5	1
B	1	1	1	2	2
C	0	2	1	5	6
D	0	1	2	3	6

- (a) Find (with proof) which team won in each of the six matches.  
 (b) Find (with proof) the scores in each of the six matches.

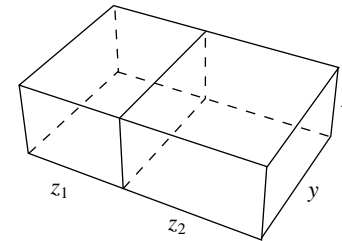
6. Two different cuboids are placed together, face-to-face, to form a large cuboid. The surface area of the large cuboid is  $\frac{2}{3}$  of the total surface area of the original two cuboids.

Prove that the lengths of the edges of the large cuboid may be labelled  $x$ ,  $y$  and  $z$ , where

$$\frac{2}{z} = \frac{1}{x} + \frac{1}{y}.$$

*First solution*

Since the two cuboids are placed together, face-to-face, to form a large cuboid, two of the edges have the same lengths. Let these common lengths be  $x$  and  $y$ , and let the other edges of the two cuboids have lengths  $z_1$  and  $z_2$ , as shown.



Now the surface area of the large cuboid is  $\frac{2}{3}$  of the total surface area of the original two cuboids. Therefore

$$2[xy + x(z_1 + z_2) + y(z_1 + z_2)] = \frac{2}{3}[2(xy + z_1x + z_1y) + 2(xy + z_2x + z_2y)]$$

so that

$$4[xy + x(z_1 + z_2) + y(z_1 + z_2)] = 3[2xy + x(z_1 + z_2) + y(z_1 + z_2)]$$

and hence

$$2xy = x(z_1 + z_2) + y(z_1 + z_2),$$

that is,

$$2xy = (x + y)(z_1 + z_2).$$

Now  $z_1 + z_2 = z$ , where  $z$  is the edge length of the large cuboid. Therefore

$$2xy = (x + y)z$$

which may be rearranged to give

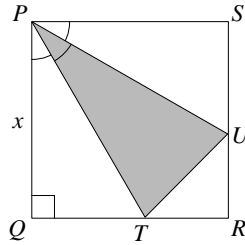
$$\begin{aligned} \frac{2}{z} &= \frac{x + y}{xy} \\ &= \frac{1}{y} + \frac{1}{x}. \end{aligned}$$

Hence the lengths of the edges of the large cuboid may be labelled  $x$ ,  $y$  and  $z$ , where

$$\frac{2}{z} = \frac{1}{x} + \frac{1}{y}.$$

Second solution

Let the sides of the square  $PQRS$  have length  $x$ .



Then in triangle  $PQT$  we have

$$\cos 30^\circ = \frac{x}{PT},$$

and hence

$$PT = \frac{x}{\cos 30^\circ}.$$

Now by symmetry  $PU = PT$  so that

$$\begin{aligned} \text{area of triangle } PTU &= \frac{1}{2}PT \times PU \sin \angle TPU \\ &= \frac{1}{2}x^2 \frac{\sin 30^\circ}{\cos^2 30^\circ}. \end{aligned}$$

Now  $\cos 30^\circ = \frac{\sqrt{3}}{2}$  and  $\sin 30^\circ = \frac{1}{2}$ . Therefore

$$\begin{aligned} \text{area of triangle } PTU &= \frac{1}{2}x^2 \times \frac{\frac{1}{2}}{\frac{3}{4}} \\ &= \frac{1}{3}x^2. \end{aligned}$$

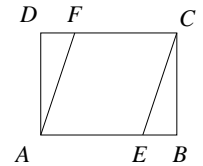
Hence the area of the triangle  $PTU$  is one third of the area of the square  $PQRS$ .

## Olympiad Hamilton Paper

All candidates must be in *School Year 10 (England and Wales), S3 (Scotland), or School Year 11 (Northern Ireland)*.

- An aquarium contains 280 tropical fish of various kinds. If 60 more clownfish were added to the aquarium, the proportion of clownfish would be doubled.  
How many clownfish are in the aquarium?
- Find the possible values of the digits  $p$  and  $q$ , given that the five-digit number ' $p543q$ ' is a multiple of 36.

- In the diagram,  $ABCD$  is a rectangle with  $AB = 16$  cm and  $BC = 12$  cm. Points  $E$  and  $F$  lie on sides  $AB$  and  $CD$  so that  $AECF$  is a rhombus.

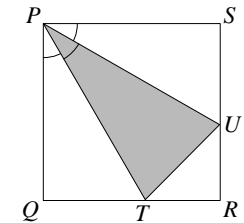


What is the length of  $EF$ ?

- Four positive integers  $a$ ,  $b$ ,  $c$  and  $d$  are such that:
  - the sum of  $a$  and  $b$  is half the sum of  $c$  and  $d$ ;
  - the sum of  $a$  and  $c$  is twice the sum of  $b$  and  $d$ ;
  - the sum of  $a$  and  $d$  is one and a half times the sum of  $b$  and  $c$ .

What is the smallest possible value of  $a + b + c + d$ ?

- The diagram shows a triangle  $PTU$  inscribed in a square  $PQRS$ . Each of the marked angles at  $P$  is equal to  $30^\circ$ .  
Prove that the area of the triangle  $PTU$  is one third of the area of the square  $PQRS$ .



- Two different cuboids are placed together, face-to-face, to form a large cuboid. The surface area of the large cuboid is  $\frac{3}{4}$  of the total surface area of the original two cuboids.  
Prove that the lengths of the edges of the large cuboid may be labelled  $x$ ,  $y$  and  $z$ , where

$$\frac{2}{z} = \frac{1}{x} + \frac{1}{y}.$$

## Olympiad Maclaurin Paper

All candidates must be in *School Year 11 (England and Wales), S4 (Scotland), or School Year 12 (Northern Ireland)*.

1. Five numbers are arranged in increasing order. As they get larger the difference between adjacent numbers doubles.  
The average of the five numbers is 11 more than the middle number. The sum of the second and fourth numbers is equal to the largest number.  
What is the largest number?

2. Miko always remembers his four-digit PIN (personal identification number) because
- it is a perfect square, and
  - it has the property that, when it is divided by 2, or 3, or 4, or 5, or 6, or 7, or 8, or 9, there is always a remainder of 1.

What is Miko's PIN?

3. Solve the simultaneous equations

$$\frac{5xy}{x+y} = 6$$

$$\frac{4xz}{x+z} = 3$$

$$\frac{3yz}{y+z} = 2.$$

4. In a trapezium  $ABCD$  the sides  $AB$  and  $DC$  are parallel and  $\angle BAD = \angle ABC < 90^\circ$ .

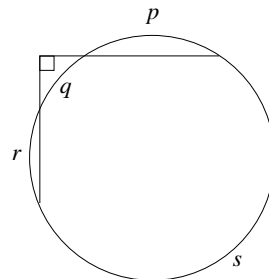
Point  $P$  lies on  $AB$  with  $\angle CPD = \angle BAD$ .

Prove that  $PC^2 + PD^2 = AB \times DC$ .

5. A lottery involves five balls being selected from a drum. Each ball has a different positive integer printed on it.

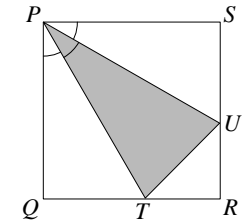
Show that, whichever five balls are selected, it is always possible to choose three of them so that the sum of the numbers on these three balls is a multiple of 3.

6. In the figure,  $p$ ,  $q$ ,  $r$  and  $s$  are the lengths of four arcs which together form the circumference of the circle. Find, in simplified form, an expression for  $s$  in terms of  $p$ ,  $q$  and  $r$ .



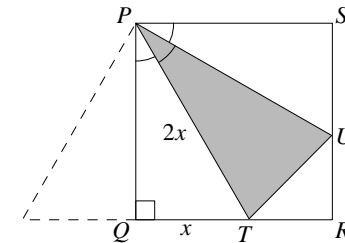
5. The diagram shows a triangle  $PTU$  inscribed in a square  $PQRS$ . Each of the marked angles at  $P$  is equal to  $30^\circ$ .

Prove that the area of the triangle  $PTU$  is one third of the area of the square  $PQRS$ .



*First solution*

Let  $QT = x$ , so that  $PT = 2x$ , since triangle  $PTQ$  is half an equilateral triangle.



Using Pythagoras' theorem in triangle  $PTQ$ , we get

$$\begin{aligned} PQ^2 &= PT^2 - QT^2 \\ &= (2x)^2 - x^2 \\ &= 3x^2 \end{aligned}$$

and hence

$$PQ = \sqrt{3}x.$$

We can now find the areas of the three unshaded right-angled triangles.

$$\begin{aligned} \text{Area of triangle } PQT &= \frac{1}{2} \times x \times \sqrt{3}x \\ &= \frac{\sqrt{3}}{2}x^2. \end{aligned}$$

Similarly,

$$\text{area of triangle } PSU = \frac{\sqrt{3}}{2}x^2.$$

Finally,

$$\begin{aligned} \text{area of triangle } TRU &= \frac{1}{2} \times (\sqrt{3}x - x) \times (\sqrt{3}x - x) \\ &= \frac{1}{2}(3x^2 - 2\sqrt{3}x^2 + x^2) \\ &= 2x^2 - \sqrt{3}x^2. \end{aligned}$$

Therefore the total unshaded area is

$$\frac{\sqrt{3}}{2}x^2 + \frac{\sqrt{3}}{2}x^2 + 2x^2 - \sqrt{3}x^2 = 2x^2.$$

However, the area of the square  $PQRS$  is  $(\sqrt{3}x)^2 = 3x^2$ . It follows that the shaded area is  $x^2$ , which is one third of the area of the square.

Once we have minimised  $b + d$ , then we automatically minimise  $a + c$ , because of equation (2), and hence minimise the sum we are interested in.

Since  $b$  and  $d$  are positive integers,  $b = 1$  and  $d = 9$  are the smallest possible values with  $d = 9b$ . From (5) and (4), we see that the corresponding values of  $c$  and  $a$  are  $c = 11$  and  $a = 9$ , both of which are also positive integers, as required.

Checking these values in equations (1) to (3), we confirm that they are valid solutions of the given equations.

Hence the smallest possible value of  $a + b + c + d$  is 30.

*Second method: elimination*

We may rearrange the three equations (1), (2) and (3) to give

$$2a + 2b = c + d \quad (7)$$

$$a + c = 2b + 2d \quad (8)$$

$$2a + 2d = 3b + 3c. \quad (9)$$

Adding (7) and (8), we get

$$3a + 2b + c = 2b + c + 3d$$

and hence

$$a = d.$$

Then (7) and (9) may be rewritten

$$2b - c + d = 0. \quad (10)$$

and

$$3b + 3c - 4d = 0. \quad (11)$$

Now adding  $3 \times (10)$  and (11), we obtain

$$9b - d = 0$$

and hence

$$d = 9b.$$

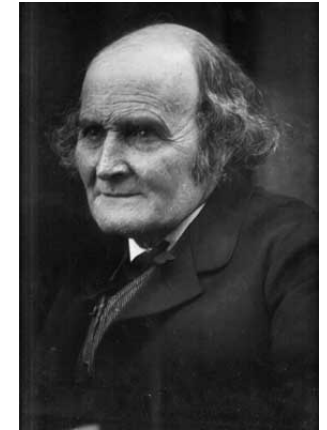
The solution now proceeds in the same way as the first method.

## Arthur Cayley

16th August 1821 – 26th January 1895

From a young age, Arthur Cayley showed great numerical flair and his mathematics teacher advised Cayley to continue his studies in this area, going against his father's wishes that he enter the family business.

Cayley did continue to study mathematics, at Trinity College, Cambridge, graduating in 1842. For four years he then taught at Cambridge and had 28 papers published in the Cambridge Mathematical Journal during this period. He then spent 14 years as a lawyer, choosing law as a profession in order to make money so he could pursue mathematics, publishing about 250 mathematical papers during this time.



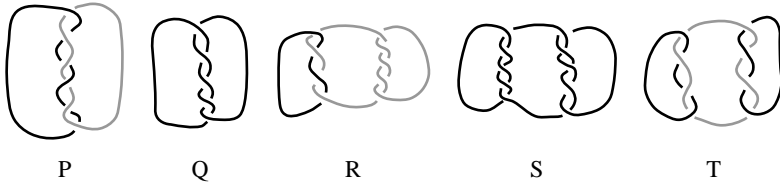
In 1863, Cayley was appointed Sadleirian professor of Pure Mathematics at Cambridge. He published over 900 papers and notes during his lifetime, covering many aspects of modern mathematics. He is remembered mainly for his work in developing the algebra of matrices and for his studies in geometry.

Around 1849 Cayley wrote a paper linking his ideas on permutations with Cauchy's. Five years later, he wrote two papers about abstract groups. At that time the only known groups were permutation groups and even these formed a radically new area, yet Cayley defined an abstract group and gave a table to display the group multiplication. He gave the 'Cayley tables' of some special permutation groups but much more significantly for the introduction of the abstract group concept, he realised that invertible matrices and quaternions form groups (with the usual operation of multiplication).

Cayley also developed the theory of algebraic invariance, and his development of  $n$ -dimensional geometry has been applied in physics to the study of the space-time continuum. His work on matrices served as a foundation for quantum mechanics, which was developed by Werner Heisenberg in 1925. Cayley also suggested that euclidean and non-euclidean geometry are special types of geometry. He united projective geometry and metrical geometry, which is dependent on sizes of angles and lengths of lines.

## Solutions to the European Kangaroo Grey Paper

1. **E** The values of the expressions are: A 9; B 11; C 191; D 209 and E 1800.
2. **B** The perimeter of the star is formed from 12 sides of the equilateral triangles and that of the hexagon from 6 sides. So the perimeter of the hexagon is  $\frac{1}{2} \times 36 = 18$  cm.
3. **E** Since  $100 = 2 \times 2 \times 5 \times 5$ , the only possible product of four *different* positive integers which equals 100 is  $1 \times 2 \times 5 \times 10$ . The sum of these integers is  $1 + 2 + 5 + 10 = 18$ .
4. **C** Observing that triangle  $PQS$  is isosceles, we have  $\angle PSQ = \frac{1}{2}(180^\circ - 12^\circ) = 84^\circ$  and hence  $\angle PSR = 180^\circ - 84^\circ = 96^\circ$ .  
Since triangle  $PRS$  is also isosceles, we have  $\angle SPR = \frac{1}{2}(180^\circ - 96^\circ) = 42^\circ$ .  
Hence  $\angle QPR = 12^\circ + 42^\circ = 54^\circ$ .
5. **A** The diagrams below show that only P, R and T are made from more than one loop.



6. **B** Firstly note that if  $n \geq 10$ , then  $n^3 \geq 10n^2$  so  $n^3$  will have at least one more digit than  $n^2$ . For all  $n < 10$ , we have  $n^2 < 100$ , so  $n^2$  has either 1 or 2 digits, but  $n^3$  has 3 digits for  $n > 4$  since  $5^3 = 125$ , so we need only consider  $n = 1, 2, 3$  or  $4$ . For  $n = 1$  and  $2$ ,  $n^2$  and  $n^3$  both have one digit; for  $n = 4$ ,  $n^2 = 16$  and  $n^3 = 64$  both have two digits. However for  $n = 3$ ,  $n^2 = 9$  has one digit while  $n^3 = 27$  has two digits.
7. **C** At least one point must be removed from each of the three horizontal lines; so at least three points need to be removed. However, removing the three points lying on either diagonal does what is required.
 

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8. **C** We shall refer to the acute-angled triangle as triangle A and the obtuse-angled triangle as triangle B. The  $120^\circ$  angle must belong to B; so the sum of its other two angles is  $60^\circ$ . Therefore the  $80^\circ$  angle must belong to A. If the  $10^\circ$  angle also belonged to A, that would make A a right-angled triangle, which it is not. So the  $10^\circ$  angle belongs to B. The final angle in B must be  $180^\circ - 120^\circ - 10^\circ = 50^\circ$ . So the  $55^\circ$  angle is in A and the final angle of A is  $180^\circ - 80^\circ - 55^\circ = 45^\circ$ .
9. **B** First, note that the middle-sized square passes through the centres of the four circles. Each side of the middle-sized square together with the edges of the outer square creates a right-angled isosceles triangle with angles of  $45^\circ$ . Thus the angles these sides make with the inner square are also  $45^\circ$ . Each side of the middle-sized square bisects the area of a circle. The inner half of that circle is made up of two shaded segments with angles of  $45^\circ$  which together are equal in area to the unshaded right-angled segment. Thus the total shaded area is exactly equal to the area of the inner square and hence equal to one-quarter of the area of the outer square.

4. Four positive integers  $a, b, c$  and  $d$  are such that:
  - the sum of  $a$  and  $b$  is half the sum of  $c$  and  $d$ ;
  - the sum of  $a$  and  $c$  is twice the sum of  $b$  and  $d$ ;
  - the sum of  $a$  and  $d$  is one and a half times the sum of  $b$  and  $c$ .

What is the smallest possible value of  $a + b + c + d$ ?

*Solution*

There are three equations here but four unknown values,  $a, b, c$  and  $d$ . Thus it is not possible just to solve the equations to find the values of  $a, b, c$  and  $d$ . What we can do is to find relationships between them and then deduce possible values of  $a, b, c$  and  $d$ .

From the given information,

$$a + b = \frac{1}{2}(c + d) \quad (1)$$

$$a + c = 2(b + d) \quad (2)$$

$$a + d = \frac{3}{2}(b + c). \quad (3)$$

We may proceed in various ways; we show two methods, substitution and elimination.

*First method: substitution*

From (1), we have

$$a = -b + \frac{1}{2}(c + d). \quad (4)$$

Substituting in (2), we get

$$-b + \frac{1}{2}(c + d) + c = 2(b + d)$$

and hence

$$\frac{3}{2}c - \frac{3}{2}d = 3b,$$

that is,

$$c - d = 2b. \quad (5)$$

Substituting from (4) in (3), we get

$$-b + \frac{1}{2}(c + d) + d = \frac{3}{2}(b + c)$$

so that

$$-c + \frac{3}{2}d = \frac{5}{2}b. \quad (6)$$

Now adding (5) and (6) we obtain

$$\frac{1}{2}d = \frac{9}{2}b,$$

and hence

$$d = 9b.$$

Using Pythagoras' theorem in triangle  $FMC$ , we have

$$CF^2 = (\frac{1}{2}AC)^2 + (\frac{1}{2}EF)^2$$

and hence

$$\left(\frac{25}{2}\right)^2 = 10^2 + (\frac{1}{2}EF)^2.$$

It follows that

$$625 = 400 + EF^2$$

and so

$$EF = 15.$$

Therefore the length of  $EF$  is 15 cm.

*Second method*

We make use of the fact that

area of rhombus  $AECF$  = area of rectangle  $ABCD$  - 2 × area of triangle  $EBC$ .

Now the area of a rhombus is half the product of its diagonals. Also, the area of triangle  $EBC$  is  $\frac{1}{2}EB \times BC$  and  $EB = 16 - \frac{25}{2} = \frac{7}{2}$ . We therefore have

$$\frac{1}{2}AC \times EF = 16 \times 12 - \frac{7}{2} \times 12.$$

Hence

$$10 \times EF = 192 - 42 = 150$$

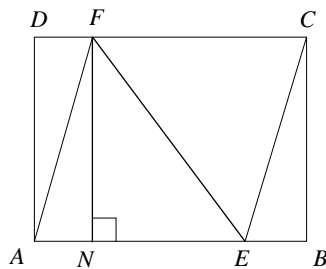
and so

$$EF = 15.$$

Therefore the length of  $EF$  is 15 cm.

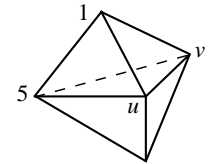
*Remark*

Another method uses Pythagoras' theorem in the right-angled triangle  $ENF$  shown below.



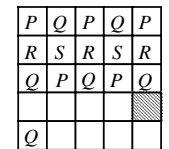
- 10. B** The first person cannot be telling the truth since if all the others are knaves, this contradicts that they are telling the truth when they say the person in front is a knave! The second person says the first is a knave so is telling the truth; he is a knight. The third says this knight is a knave so is lying; he is a knave. Continuing in this way we see that there is an alternating sequence of 13 knaves and 12 knights.

- 11. C** Let the numbers at two of the other vertices be  $u$  and  $v$ , as shown in the diagram on the right. The three faces sharing the vertex labelled with the number 1 all have the same sum. Then  $1 + v + u = 1 + 5 + u$  and so  $v = 5$ . Similarly,  $1 + v + 5 = 1 + v + u$  so  $u = 5$ . Hence the sum for each face is  $1 + 5 + 5$ , i.e. 11, and we see that the number at the bottom vertex is 1. The total of all the vertices is  $1 + 5 + 5 + 5 + 1 = 17$ .

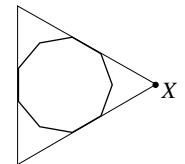


- 12. A** If we rewrite the equation we get the product  $E \times I \times G \times H \times T = T \times W \times O \times F \times O \times U \times R$ . There are 10 different letters here, so each number from 0 to 9 must be represented by one of the letters. So one letter is 0. Any product where this letter appears is 0. Hence both sides must include this letter. The only letter on both sides is  $T$ . Hence  $T = 0$  and the product  $T \times H \times R \times E \times E = 0$ .

- 13. D** It is clear that there is a unique way to complete the top three rows, as shown on the right (start in the second square of the third row). Thereafter it is possible to complete the fourth row with  $R$  and  $S$  alternating and the fifth row  $QPQPQ$ .



- 14. E** The exterior angles of a regular nonagon are  $360^\circ \div 9 = 40^\circ$ , whence the interior angles are  $180^\circ - 40^\circ = 140^\circ$ . In the arrowhead quadrilateral whose rightmost vertex is  $X$ , three of the angles are  $40^\circ, 40^\circ$  and  $360^\circ - 140^\circ = 220^\circ$  and these add up to  $300^\circ$ . So the angle at  $X$  is  $60^\circ$ .

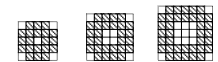


[It is now possible to see that the entire nonagon can fit neatly inside an equilateral triangle and so the angle at  $X$  is  $60^\circ$ .]

- 15. D** One way to proceed is to regard the pattern as four arms, each two squares wide, with four corner pieces of three squares each. So for the  $n$ th pattern, we have  $4 \times 2 \times n + 4 \times 3 = 8n + 12$ . For  $n = 10$ , we need  $8 \times 10 + 12$ , i.e. 92 squares.



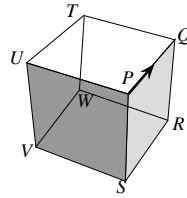
[Alternatively, it is possible to see the pattern as a complete square with the four corners and a central square removed. So for the  $n$ th pattern, we have a complete  $(n + 4)(n + 4)$  square with the four corners and a central  $n \times n$  square removed. Hence the number of squares is  $(n + 4)^2 - n^2 - 4 = 8n + 12$ .



- 16. C** The digits 1 and 3 will always be followed by the digit 2. The digit 2 can be followed by either 1 or 3. Hence the digit 2 appears exactly five times in a ten-digit number, in alternate positions.

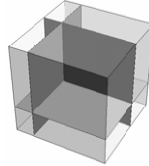
If the first digit is 2, then in each even position we have two choices, 1 or 3. This gives  $2 \times 2 \times 2 \times 2 \times 2 = 32$  possibilities. Otherwise, the second digit is 2 and in each odd position we have two choices. So again there are 32 possibilities, making a total of 64.

17. **B** At  $Q$  the ant can choose first to go left to  $T$ , then right to  $W$ . Otherwise, at  $Q$  he can go right to  $R$  and then left to  $W$ . Note that  $W$  is the corner diagonally opposite to  $P$  and is reached by either route after three edges (and no fewer). So after exactly three more edges, the ant must reach the corner opposite  $W$ , that is,  $P$ .



18. **A** The difference between  $\frac{1}{3}$  and  $\frac{1}{5}$  is  $\frac{1}{3} - \frac{1}{5} = \frac{2}{15}$ . This section of the number line is divided into 16 intervals, each of length  $\frac{2}{15} \div 16 = \frac{1}{120}$ . The difference between  $\frac{1}{4}$  and  $\frac{1}{5}$  is  $\frac{1}{4} - \frac{1}{5} = \frac{1}{20} = \frac{6}{120}$ , and hence  $\frac{1}{4}$  is six smaller intervals from  $\frac{1}{5}$ , at point  $a$ .

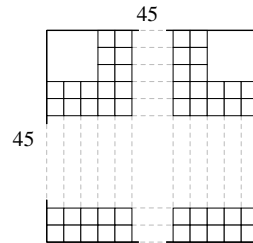
19. **D** After the cuts, eight smaller cuboids are formed and so we can conclude that the cuts are parallel to the faces of the large cube. Each of the smaller cuboids has three matching pairs of faces, one on the outside of the large cube and one inside. So the total surface area of the smaller cuboids is twice the surface area of the cube.



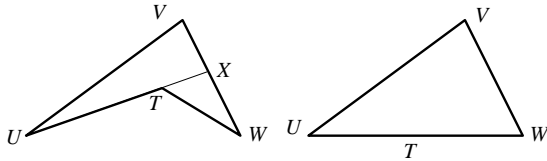
20. **C** Let the smallest prime factor of  $N$  be  $p$ , whence the second largest factor, and the highest factor Tina wrote down, is  $\frac{N}{p}$ . Now we have  $\frac{N}{p} = 45p$  whence  $N = 45p^2$ .

Since  $N$  is a multiple of 45, it has prime factors of 3 and 5 and, because  $p$  is the smallest prime factor of  $N$ , we can conclude that  $p$  can be only 2 or 3. Hence either  $N = 45 \times 2^2 = 180$  or  $N = 45 \times 3^2 = 405$ .

21. **B** It is clear that a  $44 \times 44$  square would be too small to accommodate 2009 squares, because  $44 \times 44 = 1936$ . Now  $45 \times 45 = 2025$ , which is 16 squares too many. However, if we use 18 squares as shown in the diagram to form two  $3 \times 3$  squares we get a total of 2009.



22. **E** Consider first a general case where in a quadrilateral  $TUVW$  the interior angle at  $T$  is more than  $180^\circ$ . Let  $UT$  be extended to meet  $VW$  at  $X$ .



Then, by the triangle equality applied to the triangles  $UXV$  and  $TWX$ , it follows that

$$UX < UV + VX$$

and

$$TW < TX + XW.$$

Adding these inequalities gives that is

$$UX + TW < UV + VX + TX + XW,$$

$UT + TX + TW < UV + VX + TX + VW.$

Hence

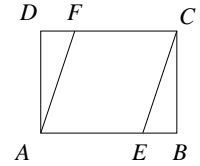
$$UT + TW < UV + VX + XW,$$

that is

$$UT + TW < UV + VW.$$

In the case where the angle at  $T$  is  $180^\circ$ ,  $UVW$  is a triangle and by the triangle equality  $UW < UV + VW$ , that is, again,  $UT + TW < UV + VW$ .

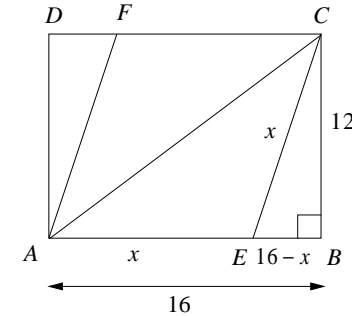
3. In the diagram,  $ABCD$  is a rectangle with  $AB = 16$  cm and  $BC = 12$  cm. Points  $E$  and  $F$  lie on sides  $AB$  and  $CD$  so that  $AECF$  is a rhombus.



What is the length of  $EF$ ?

*Solution*

Let the sides of the rhombus  $AECF$  have length  $x$  cm. Hence  $AE = x$  and  $EB = 16 - x$ . Since  $ABCD$  is a rectangle, angle  $EBC$  is a right angle.



Using Pythagoras' theorem in triangle  $ABC$ , we have  $AC^2 = 16^2 + 12^2 = 400$ , so that  $AC = 20$  cm.

Using Pythagoras' theorem in triangle  $EBC$ , we have

$$EC^2 = CB^2 + EB^2$$

and hence

$$x^2 = 12^2 + (16 - x)^2,$$

which can be rearranged to give

$$x^2 = 144 + 256 - 32x + x^2.$$

It follows that

$$32x = 400$$

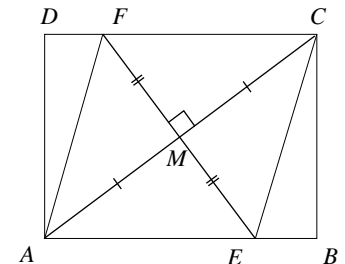
and so

$$x = \frac{25}{2}.$$

We may now proceed in various ways; we show two different methods.

*First method*

Let  $M$  be the point of intersection of the diagonals  $AC$  and  $EF$  of  $AECF$ . Since  $AECF$  is a rhombus, angle  $FMC$  is a right angle and  $M$  is the mid-point of both  $AC$  and  $EF$ .



## Solutions to the Olympiad Hamilton Paper

1. An aquarium contains 280 tropical fish of various kinds. If 60 more clownfish were added to the aquarium, the proportion of clownfish would be doubled.

How many clownfish are in the aquarium?

*Solution*

Let there be  $x$  clownfish in the aquarium.

If 60 clownfish are added there are  $x + 60$  clownfish and 340 tropical fish in total.

Since the proportion of clownfish is then doubled, we have

$$2 \times \frac{x}{280} = \frac{x + 60}{340}.$$

Multiplying both sides by 20, we get

$$\frac{x}{7} = \frac{x + 60}{17}$$

and hence

$$17x = 7(x + 60).$$

It follows that  $x = 42$  and thus there are 42 clownfish in the aquarium.

2. Find the possible values of the digits  $p$  and  $q$ , given that the five-digit number 'p543q' is a multiple of 36.

*Solution*

Since 'p543q' is a multiple of 36 it is a multiple of both 9 and 4.

The sum of the digits of a multiple of 9 is also a multiple of 9, hence  $p + 5 + 4 + 3 + q$  is a multiple of 9. But  $5 + 4 + 3 = 12$  and each of  $p$  and  $q$  is a single digit, so that  $p + q = 6$  and  $p + q = 15$  are the only possibilities.

Since 'p543q' is a multiple of 4 and 'p5400' is always divisible by 4, it follows that '3q' is divisible by 4. The only possible values for '3q' are 32 and 36, so that  $q = 2$  or  $q = 6$ .

If  $q = 2$ , then  $p + q = 15$  is not possible since  $p$  is a single digit. Hence  $p + q = 6$  and so  $p = 4$ .

If  $q = 6$ , then  $p + q = 6$  is not possible since 'p543q' is a five-digit number and therefore the digit  $p$  cannot be zero. Hence  $p + q = 15$  and so  $p = 9$ .

Therefore  $p = 4, q = 2$  and  $p = 9, q = 6$  are the only possible values of the digits  $p$  and  $q$ .

Thus if the interior angle at a given vertex of a quadrilateral is  $\theta$  where  $\theta \geq 180^\circ$  the sum of the lengths of the sides adjacent to this vertex is less than the sum of the lengths of the other two sides.

[The above argument appears as Proposition 21 in Book 1 in the ancient treatise on geometry, *The Elements*, by the Greek mathematician Euclid. There is an excellent online version (at <http://tinyurl.com/6neb9e>) which readers might like to consult.]

In this case, we have that  $QP + PS = 2006 + 2009 = 2008 + 2007 = QR + RS$ , from which it immediately follows by the above result that the interior angles at  $P$  and at  $R$  must each be less than  $180^\circ$ . Also  $PS + SR = 2009 + 2007 > 2006 + 2008 = PQ + QR$ . Again it follows from the above result that the interior angle at  $S$  must be less than  $180^\circ$ . However the angle at  $Q$  need not be less than  $180^\circ$ . For example, this occurs in the case where  $PR = 4000$  and  $Q$  is an interior point of the triangle  $PRS$ .

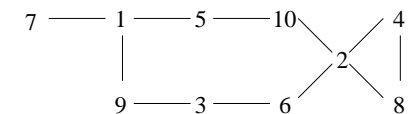
23. **D** Suppose that the triangle and the square are placed so that the area of overlap is as large as possible. The area of the square is  $36 \text{ cm}^2$ . The area of overlap is  $\frac{2}{3}$  of this, namely  $24 \text{ cm}^2$ . This is 60% of the area of the triangle. So the area of the triangle is  $\frac{10}{6} \times 24 = 40 \text{ cm}^2$ .
24. **D** Suppose it is possible to make a list of all ten numbers.

The number 7 must be at one end and must be next to 1 since 7 has no other factors or multiples under 10. Without loss of generality we can assume 7 is the first number, followed by 1.

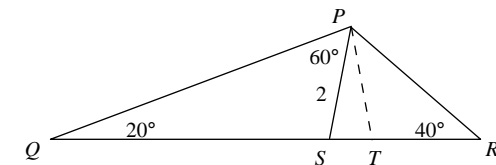
The number 5 only has two possible adjacent numbers, 1 and 10. The same is true for 9 which can only be next to 1 or 3. Hence either we must start with 7, 1, 5, 10 and end with 9; or we start with 7, 1, 9, 3 and end in 5. Either way this means that 1 cannot be next to any other numbers.

The diagram below shows the only possible connections that can be used. It is clearly impossible to link all ten numbers together without using 2 twice. If the sequence starts 7, 1, 5, 10 then the only possibility after 10 is 2 but the only possibility before 6 is 2 which means 2 has to appear twice; or if the sequence starts 7, 1, 9, 3, 6 then the only possibility after 6 is 2 and the only possibility before 10 is 2 so 2 is used twice.

However, the diagram suggests a possible list of nine numbers:  
6, 3, 9, 1, 5, 10, 2, 4, 8.



25. **C** Consider triangle  $PQR$ . Using the angle sum of a triangle,  $\angle QPR = 180^\circ - 40^\circ - 20^\circ = 120^\circ$ . Thus  $\angle QPS = 60^\circ$  and, by using the exterior angle property,  $\angle PST = 80^\circ$ .

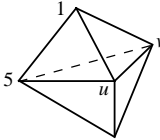


Let  $T$  be the point on  $QR$  such that  $QT = QP$  which means that triangle  $PQT$  is isosceles. Hence  $\angle QTP = \frac{1}{2}(180^\circ - 20^\circ) = 80^\circ$ . Thus triangle  $PST$  is also isosceles and therefore  $PT = 2$ .

Since  $\angle QTP = 80^\circ$ , it follows from the exterior angle property applied to triangle  $PTR$  that  $\angle TPR = 40^\circ$ . Thus this triangle is also isosceles and so  $TR = TP$  and so  $QR$  is 2 units longer than  $PQ$ .

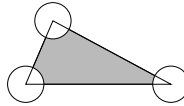
## Solutions to the European Kangaroo Pink Paper

1. **C** Kanga divides the other 2008 participants in the ratio 1:3, so has  $2008 \div 4 = 502$  participants ahead of her. She comes in 503rd.
2. **A** When multiplying the fractions together, the denominator of each fraction, apart from the last, cancels with the numerator of the next fraction. We are left with the numerator 1 from the first fraction and the denominator 10 from the last, which gives  $\frac{1}{10}$  of 1000, i.e. 100.
3. **C** Let the numbers at two of the other vertices be  $u$  and  $v$ , as shown in the diagram on the right. The three faces sharing the vertex labelled with the number 1 all have the same sum. Then  $1 + v + u = 1 + 5 + u$  and so  $v = 5$ . Similarly,  $1 + v + 5 = 1 + v + u$  so  $u = 5$ . Hence the sum for each face is  $1 + 5 + 5$ , i.e. 11, and we see that the number at the bottom vertex is 1. The total of all the vertices is  $1 + 5 + 5 + 5 + 1 = 17$ .

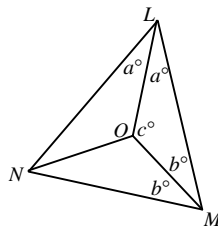


4. **B** Firstly note that if  $n \geq 10$ , then  $n^3 \geq 10n^2$  so  $n^3$  will have at least one more digit than  $n^2$ . For all  $n < 10$ , we have  $n^2 < 100$ , so  $n^2$  has either 1 or 2 digits, but  $n^3$  has 3 digits for  $n > 4$  since  $5^3 = 125$ , so we need only consider  $n = 1, 2, 3$  or 4. For  $n = 1$  and 2,  $n^2$  and  $n^3$  both have one digit; for  $n = 4$ ,  $n^2 = 16$  and  $n^3 = 64$  both have two digits. However for  $n = 3$ ,  $n^2 = 9$  has one digit while  $n^3 = 27$  has two digits.

5. **B** The three angles of the triangle add to  $180^\circ$ , so the combined area of the three sectors of the circles that are inside the triangle add up to half a circle with area  $\frac{1}{2} \times \pi \times 2^2 = \frac{4\pi}{2} = 2\pi$ . So the grey area is  $(80 - 2\pi) \text{ cm}^2$ .



6. **E** If we let the fifth number be  $a$ , then the sixth number is  $6 + a = 15$ , so  $a = 9$ . The seventh number is the sum of the fifth and sixth numbers,  $9 + 15 = 24$ .
7. **B** Let  $\angle OLM = \angle OLN = a^\circ$ ,  $\angle OML = \angle OMN = b^\circ$  and  $\angle LOM = c^\circ$ . Angles in a triangle add up to  $180^\circ$ , so from  $\triangle LMN$ ,  $2a^\circ + 2b^\circ + 68^\circ = 180^\circ$  which gives  $2(a^\circ + b^\circ) = 112^\circ$  i.e.  $a + b = 56$ . Also, from  $\triangle LOM$ ,  $a^\circ + b^\circ + c^\circ = 180^\circ$  and so  $c = 180 - (a + b) = 180 - 56 = 124$ .



8. **E** Mary's average over four tests is 4 marks, so she has scored 16 marks in total. The first four options can give a total of 16 as follows:
- A Mary achieved a mark of 4 out of 5 in each test:  $4 + 4 + 4 + 4 = 16$
- B Mary achieved a mark of 4 out of 5 twice:  $4 + 4 + 3 + 5 = 16$
- C Mary achieved a mark of 1 out of 5 once:  $1 + 5 + 5 + 5 = 16$
- D Mary achieved a mark of 3 out of 5 twice:  $3 + 3 + 5 + 5 = 16$

Of the five cases given in the question, only the last one makes it impossible to score 16 marks since she has scored 9 marks from three tests, and would need 7 marks from the fourth test (which is only out of 5).

We observe that the row for C now means that  $x + y + z = 5$  (which agrees with the column for C).

From the column for A we see that  $z$  is at most 1, since the total in that column is 1. Similarly, from the row for D, we see that  $y$  is at most 3, and from the row for B we see that  $x$  is at most 1 since  $t$  is at least 1.

But we have  $x + y + z = 5$ , so that the only possibilities are  $x = 1, y = 3$  and  $z = 1$ . It follows that  $t = 1$ .

Therefore the table is:

		Goals against				
		A	B	C	D	All
Goals for	A	–		2		5
	B		–	1	1	2
	C	1	1	–	3	5
	D			3	–	3
	All	1	2	6	6	15

We may now complete the table by, for example, first noting that all other entries in the column for A are 0, and then filling in the rows from the bottom.

		Goals against				
		A	B	C	D	All
Goals for	A	–	1	2	2	5
	B	0	–	1	1	2
	C	1	1	–	3	5
	D	0	0	3	–	3
	All	1	2	6	6	15

In summary, the scores in each match were as follows:

A beat B	1 – 0
A beat C	2 – 1
A beat D	2 – 0
B drew with C	1 – 1
B beat D	1 – 0
C drew with D	3 – 3

6. Teams A, B, C and D competed against each other once. The results table was as follows:

Team	Win	Draw	Loss	Goals for	Goals against
A	3	0	0	5	1
B	1	1	1	2	2
C	0	2	1	5	6
D	0	1	2	3	6

- (a) Find (with proof) which team won in each of the six matches.  
 (b) Find (with proof) the scores in each of the six matches.

*Solution*

(a) Team A won all three games and so beat teams B, C and D.

Of the three games that team C played, the one that was lost can only have been against team A. Therefore team C drew against teams B and D.

If we consider the three games that team B played, the game against team A was lost, the game against team C was a draw and so the remaining game, that team B won, was against team D.

In summary:

- A beat B, A beat C, A beat D;
- B drew with C, B beat D; and
- C drew with D.

- (b) Consider the following table in which the rows give the number of goals scored for each team and the columns give the number of goals against each team.

		Goals against				
		A	B	C	D	All
Goals for	A	–		$z + 1$		5
	B		–	$x$	$t$	2
	C	$z$	$x$	–	$y$	5
	D			$y$	–	3
	All	1	2	6	6	15

We have let the number of goals scored by team C against team B be  $x$ , so that the number of goals scored by team B against team C is also  $x$ , since their match was a draw. Similarly, we have let the number of goals scored by team C against team D be  $y$ , so that this is also the number scored by team D against team C.

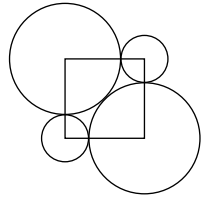
Furthermore, we have let the number of goals scored by team C against team A be  $z$ , so that the number of goals scored by team A against team C is  $z + 1$  since the difference between the number of goals scored and conceded by team C is 1.

Finally, we have let the number of goals scored by team B against team D be  $t$ . Then  $t$  is at least 1 since team B beat team D.

9. **B** The first person cannot be telling the truth since if all the others are knaves, this contradicts that they are telling the truth when they say the person in front is a knave! The second person says the first is a knave so is telling the truth; he is a knight. The third says this knight is a knave so is lying; he is a knave. Continuing in this way we see that there is an alternating sequence of 13 knaves and 12 knights.

10. **E**  $3 \oplus 5 = 3 \times 5 + 3 + 5 = 23$  and  $2 \oplus x = 2x + 2 + x = 3x + 2$ . These are equal, so  $3x + 2 = 23$ , i.e.  $x = 7$ .

11. **A** Let  $R$  be the radius of each of the larger circles. The sides of the square are equal to  $R + 1$ , the sum of the two radii. The diagonal of the square is  $2R$ . By Pythagoras,  $(R + 1)^2 + (R + 1)^2 = (2R)^2$ . Simplifying gives  $2(R + 1)^2 = 4R^2$ , i.e.  $(R + 1)^2 = 2R^2$ , so  $R + 1 = \sqrt{2}R$  [ $-\sqrt{2}R$  is not possible since  $R + 1 > 0$ ]. Therefore  $(\sqrt{2} - 1)R = 1$ . Hence  $R = \frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1$ .



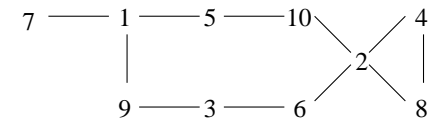
12. **C** We require that  $9 < \sqrt{n} < 11$ , or, equivalently, that  $81 < n < 121$ . Hence the possible integer values for  $n$  are the 39 values  $n = 82, 83, \dots, 119, 120$ .

13. **D** Suppose it is possible to make a list of all ten numbers. The number 7 must be at one end and must be next to 1 since 7 has no other factors or multiples under 10. Without loss of generality we can assume 7 is the first number, followed by 1.

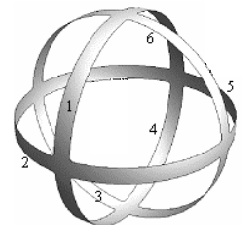
The number 5 only has two possible adjacent numbers, 1 and 10. The same is true for 9 which can only be next to 1 or 3. Hence either we must start with 7, 1, 5, 10 and end with 9; or we start with 7, 1, 9, 3 and end in 5. Either way this means that 1 cannot be next to any other numbers.

The diagram below shows the only possible connections that can be used. It is clearly impossible to link all ten numbers together without using 2 twice. If the sequence starts 7, 1, 5, 10 then the only possibility after 10 is 2 but the only possibility before 6 is 2 which means 2 has to appear twice; or if the sequence starts 7, 1, 9, 3, 6 then the only possibility after 6 is 2 and the only possibility before 10 is 2 so 2 is used twice.

However, the diagram suggests a possible list of nine numbers: 6, 3, 9, 1, 5, 10, 2, 4, 8.



14. **A** We may suppose that the ant starts at the top. The diagram shows the six quarter-circles that she travels through before arriving back at the top.



15. **C** The options are all of the form  $1 + \frac{1}{n}$  so we need to find  $n$  such that  $1 + \frac{1}{20008} < 1 + \frac{1}{n} < 1 + \frac{1}{2008}$ . This means  $2008 < n < 20008$ . The choices for  $n$  are 100, 1000, 10000, 100000 and 1000000 but only 10000 satisfies the inequalities. We get  $1 + \frac{1}{10000} = 1.0001$ .
16. **E** By comparing  $b$  and  $c$  first, we have  $b = 8^8 = (2^3)^8 = 2^{24} = (2^2)^{12} = 4^{12} > 3^{11} = c$  so  $c < b$ . But also  $b = 2^{24} < 2^{25} = a$  so  $b < a$ . Together these give  $c < b < a$ .
17. **C** The digits 1 and 3 will always be followed by the digit 2. The digit 2 can be followed by either 1 or 3. Hence the digit 2 appears exactly five times in a ten-digit number, in alternate positions.  
If the first digit is 2, then in each even position we have two choices, 1 or 3. This gives  $2 \times 2 \times 2 \times 2 \times 2 = 32$  possibilities. Otherwise, the second digit is 2 and in each odd position we have two choices. So again there are 32 possibilities, making a total of 64.
18. **C** Any three positive integers that multiply to make 2009 would create viable cuboids. The prime factors of 2009 are  $7 \times 7 \times 41$ , so the options are  $1 \times 1 \times 2009$ ,  $1 \times 7 \times 287$ ,  $1 \times 41 \times 49$  and  $7 \times 7 \times 41$ . The first three cuboids all have two faces which each require 2009 stickers ( $1 \times 2009$ ,  $7 \times 287$  and  $41 \times 49$  respectively) so Roo cannot cover them. The last cuboid has surface area  $2 \times (7 \times 7 + 7 \times 41 + 41 \times 7) = 1246$ , leaving  $2009 - 1246 = 763$  stickers left over.
19. **C** Let the smallest prime factor of  $N$  be  $p$ , whence the second largest factor, and the highest factor Tina wrote down, is  $\frac{N}{p}$ . Now we have  $\frac{N}{p} = 45p$  whence  $N = 45p^2$ .  
Since  $N$  is a multiple of 45, it has prime factors of 3 and 5 and, because  $p$  is the smallest prime factor of  $N$ , we can conclude that  $p$  can be only 2 or 3. Hence either  $N = 45 \times 2^2 = 180$  or  $N = 45 \times 3^2 = 405$ .
20. **B** Each fruit can have at most two fruits next to it but each type of fruit must be next to three other types of fruit so there are at least two of every fruit. This means there are at least 8 fruits in total. In fact 8 are sufficient, as shown in the arrangement OABPOBAP (O or Orange, P for Peach, A for Apple, B for Banana).
21. **A** There are four rows and four columns, so we need eight different sums. The smallest eight sums (if possible) would be 0, 1, 2, 3, ..., 7. Since each draught is counted towards the sum of a row and the sum of a column, we would need  $\frac{1}{2}(0 + 1 + 2 + \dots + 7) = 14$  draughts. The diagram shows it is possible to place 14 draughts on the board to create the eight smallest sums (the numbers in the cells represent how many draughts there are in each cell, and the column and row totals are shown).
- |   |   |   |   |   |
|---|---|---|---|---|
|   |   |   |   | 0 |
| 1 | 1 |   |   | 2 |
|   | 1 | 4 |   | 5 |
|   | 1 |   | 6 | 7 |
|   | 1 | 3 | 4 | 6 |
22. **B** We make use of two key facts. First, we have the factorization  $n^2 - 1 = (n - 1)(n + 1)$ . Second, when a square number is factorized, each prime factor appears an even number of times.  
Now  $2^2 - 1 = 1 \times 3$ . We next get a prime factor 3 with  $4^2 - 1 = 3 \times 5$ . We next get a factor 5 with  $6^2 - 1 = 5 \times 7$ . We next get a factor 7 with  $8^2 - 1 = 7 \times 9$ . As  $9 = 3^2$ , it does not require any further factors. Hence we need  $n \geq 8$ . Checking the product with  $n = 8$ , we get  $(2^2 - 1)(3^2 - 1)(4^2 - 1)(5^2 - 1)(6^2 - 1)(7^2 - 1)(8^2 - 1) = 1 \times 3 \times 2 \times 4 \times 3 \times 5 \times 4 \times 6 \times 5 \times 7 \times 6 \times 8 \times 7 \times 9 = 2 \times 8 \times 3 \times 3 \times 4 \times 4 \times 5 \times 5 \times 6 \times 6 \times 7 \times 7 \times 9 = 4 \times 4 \times 3 \times 3 \times 4 \times 4 \times 5 \times 5 \times 6 \times 6 \times 7 \times 7 \times 3 \times 3 = (4 \times 3 \times 4 \times 5 \times 6 \times 7 \times 3)^2$ .  
So in fact  $n = 8$  is sufficient, and is thus the minimum.

5. Two candles are the same height. The first takes 10 hours to burn completely whilst the second takes 8 hours to burn completely.  
Both candles are lit at midday. At what time is the height of the first candle twice the height of the second candle?

*Solution*

Let the initial height of each candle be  $h$  cm. In one hour the first candle will burn  $\frac{h}{10}$  cm and the second candle will burn  $\frac{h}{8}$  cm. Thus in  $t$  hours, the candles will burn

$$\frac{ht}{10} \text{ cm} \quad \text{and} \quad \frac{ht}{8} \text{ cm},$$

respectively.

If both candles are lit at midday, then  $t$  hours after midday the heights of the first and second candles will be

$$\left(h - \frac{ht}{10}\right) \text{ cm} \quad \text{and} \quad \left(h - \frac{ht}{8}\right) \text{ cm},$$

respectively.

We are asked to find the time at which the height of the first candle is twice the height of the second candle. We therefore need to find the value of  $t$  such that

$$h - \frac{ht}{10} = 2\left(h - \frac{ht}{8}\right).$$

We may divide every term by  $h$ , since we know that  $h$  is not zero, and expand the brackets to obtain the equation

$$1 - \frac{t}{10} = 2 - \frac{t}{4}.$$

Multiplying both sides by 20, we get

$$20 - 2t = 40 - 5t,$$

and so

$$t = \frac{20}{3} = 6\frac{2}{3}.$$

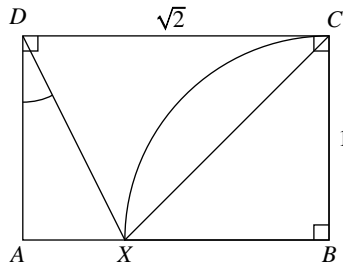
Hence the height of the first candle is twice that of the second after 6 hours and 40 minutes, in other words, this happens at 18:40.

4. In the rectangle  $ABCD$ , the side  $AB$  has length  $\sqrt{2}$  and the side  $AD$  has length 1. Let the circle with centre  $B$  and passing through  $C$  meet  $AB$  at  $X$ .

Find  $\angle ADX$  (in degrees).

*Solution*

We begin with a diagram showing the information given in the question. We have used the fact that  $ABCD$  is a rectangle, so that  $BC = AD = 1$  and  $DC = AB = \sqrt{2}$ , and angles  $ABC$ ,  $BCD$  and  $CDA$  are right angles.



Since  $BX$  and  $BC$  are both radii of the circle,  $BX$  also has length 1. This means that triangle  $XBC$  is isosceles and so  $\angle BXC = \angle BCX$ .

Furthermore, since  $\angle ABC$  is a right angle,  $\angle BXC$  and  $\angle BCX$  are both equal to  $45^\circ$ .

From the fact that  $\angle BCD$  is a right angle, it follows that  $\angle XCD = 90^\circ - 45^\circ = 45^\circ$ .

We may use Pythagoras' theorem in triangle  $XBC$  to obtain

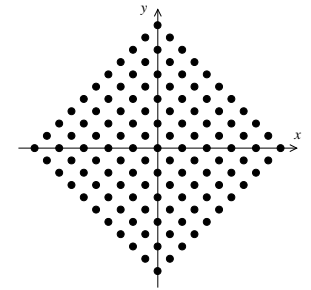
$$\begin{aligned} XC^2 &= BX^2 + BC^2 \\ &= 1^2 + 1^2 \\ &= 2 \end{aligned}$$

and so  $XC = \sqrt{2}$ .

We are given that  $DC$  also has length  $\sqrt{2}$  and so triangle  $XCD$  is isosceles. This means that  $\angle CXD$  and  $\angle CDX$  are equal, and so each is equal to  $(180^\circ - 45^\circ) \div 2 = 67\frac{1}{2}^\circ$ .

Lastly, we use the fact that  $\angle CDA$  is a right angle to conclude that  $\angle ADX = 90^\circ - 67\frac{1}{2}^\circ = 22\frac{1}{2}^\circ$ .

23. **B** Consider the kangaroo's starting position as the origin of coordinate axes, with East and North being the positive  $x$  and  $y$  directions, respectively, and one metre being one unit along the axes. We begin by considering the first quadrant. If the kangaroo's end point has coordinates  $(a, b)$ , then  $a$  and  $b$  must be integers. Also, after 10 jumps, it must be that  $a + b \leq 10$ . Hence his end points are bounded by the right-angled triangle with vertices at  $(10, 0)$ ,  $(0, 10)$  and  $(0, 0)$ . He can finish at any point on the hypotenuse of this triangle since all these points satisfy  $a + b = 10$  and so can be reached by  $a$  jumps East and  $b$  jumps North. But he can only end up at a point  $(a, b)$  on the other two edges or inside the triangle if  $a + b$  is even. (He can certainly reach all such points in  $a + b \leq 10$  jumps, and if  $a + b$  is even, with  $a + b < 10$ , he can jump away and back again using up 2 jumps, and can repeat this until he has made 10 jumps, and so end up at  $(a, b)$ .)



By symmetry we see that the possible end points form a square of side 11, and so there are 121 of them, as shown in the diagram.

24. **C** The following seven pairs add to 16 so at least one of each pair must be removed:  $(1, 15)$ ,  $(2, 14)$ ,  $(3, 13)$ ,  $(4, 12)$ ,  $(5, 11)$ ,  $(6, 10)$ ,  $(7, 9)$ .

If removing these seven is sufficient, then we would be left with 8, 16 and seven others.

But	$16 + 9 = 25$	so we must remove 9 (and keep its partner 7).
	$7 + 2 = 9$	so we must remove 2 and keep 14.
	$14 + 11 = 25$	so we must remove 11 and keep 5.
	$5 + 4 = 9$	so we must remove 4 and keep 12.
	$12 + 13 = 25$	so we must remove 13 and keep 3.
	$3 + 1 = 4$	so we must remove 1 and keep 15.
	$15 + 10 = 25$	so we must remove 10 and keep 6.

But we have kept 3 and 6 which add to 9.

Hence it is not sufficient to remove only seven. If we remove the number 6, we obtain a set which satisfies the condition:  $\{8, 16, 7, 14, 5, 12, 3, 15\}$  or in ascending order  $\{3, 5, 7, 8, 12, 14, 15, 16\}$ . Hence eight is the smallest number of numbers that may be removed.

25. **D** The 1-digit primes are 2, 3, 5 and 7. Any two of these make a two-digit number which is strange if the result is a prime. A 2-digit prime cannot end in 2 or 5, and we can also exclude 27, 57 because they are divisible by 3; also 33 and 77 are divisible by 11. This leaves four 2-digit strange primes: 23, 53, 73, 37.
- A 3-digit strange prime will be the concatenation of two 2-digit strange primes where the last digit of the first prime is the first digit of the second prime. The possibilities are: 23 and 37 to make 237; 53 and 37 to make 537; 73 and 37 to make 737; 37 and 73 to make 373. However, 237 and 537 are divisible by 3, and 737 is divisible by 11. This leaves only one 3-digit strange prime, 373. Therefore a 4-digit strange prime can only begin with 373 (making the second digit 7) and end with 373 (making the second digit 3) which is impossible. Since there are no 4-digit primes, we cannot make a strange prime with more than 4 digits. Hence there are nine strange primes: 2, 3, 5, 7, 23, 37, 53, 73, 373.

## Solutions to the Olympiad Cayley Paper

1. An aquarium contains 280 tropical fish of various kinds. If 60 more clownfish were added to the aquarium, the proportion of clownfish would be doubled. How many clownfish are in the aquarium?

### Solution

Let there be  $x$  clownfish in the aquarium.

If 60 clownfish are added there are  $x + 60$  clownfish and 340 tropical fish in total.

Since the proportion of clownfish is then doubled, we have

$$2 \times \frac{x}{280} = \frac{x + 60}{340}.$$

Multiplying both sides by 20 we get

$$\frac{x}{7} = \frac{x + 60}{17}$$

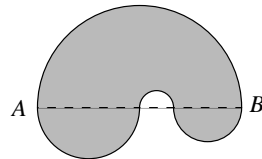
and hence

$$17x = 7(x + 60).$$

It follows that  $x = 42$  and thus there are 42 clownfish in the aquarium.

2. The boundary of the shaded figure consists of four semicircular arcs whose radii are all different. The centre of each arc lies on the line  $AB$ , which is 10 cm long.

What is the length of the perimeter of the figure?



### Solution

The centre of the large semicircular arc lies on  $AB$ , so we know that  $AB$  is a diameter of the large semicircle. But  $AB$  is 10 cm long, so the radius of the large semicircle is 5 cm.

Let the radii of the other three semicircles be  $r_1$  cm,  $r_2$  cm and  $r_3$  cm. The centres of these arcs also lie on  $AB$ , so the sum of their diameters is equal to the length of  $AB$ . It follows that  $2r_1 + 2r_2 + 2r_3 = 10$  and hence  $r_1 + r_2 + r_3 = 5$ .

Now the lengths, in cm, of the semicircular arcs are  $5\pi$ ,  $\pi r_1$ ,  $\pi r_2$  and  $\pi r_3$ . Therefore the perimeter of the figure has length, in cm,

$$\begin{aligned} 5\pi + \pi r_1 + \pi r_2 + \pi r_3 &= \pi(5 + r_1 + r_2 + r_3) \\ &= \pi(5 + 5) \\ &= 10\pi. \end{aligned}$$

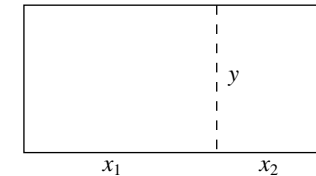
Hence the perimeter of the figure has length  $10\pi$  cm.

3. Two different rectangles are placed together, edge-to-edge, to form a large rectangle. The length of the perimeter of the large rectangle is  $\frac{2}{3}$  of the total perimeter of the original two rectangles.

Prove that the final rectangle is in fact a square.

### First solution

Since the smaller rectangles are placed together edge-to-edge, they have a side length in common. Let this side have length  $y$  and let the other sides have lengths  $x_1$  and  $x_2$  as shown.



The perimeters of the smaller rectangles are  $2x_1 + 2y$  and  $2x_2 + 2y$ , so the total perimeter of the two smaller rectangles is  $2x_1 + 2x_2 + 4y$ .

The perimeter of the large rectangle is  $2(x_1 + x_2) + 2y = 2x_1 + 2x_2 + 2y$ .

We are given that the length of the perimeter of the large rectangle is  $\frac{2}{3}$  of the total perimeter of the two original rectangles. Hence we may form the equation

$$2x_1 + 2x_2 + 2y = \frac{2}{3}(2x_1 + 2x_2 + 4y).$$

We may simplify this equation by multiplying both sides by 3 and expanding the brackets, to obtain

$$6x_1 + 6x_2 + 6y = 4x_1 + 4x_2 + 8y,$$

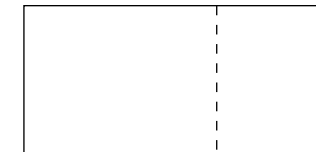
which simplifies to

$$x_1 + x_2 = y.$$

This means that the length and width of the large rectangle are the same. In other words, the rectangle is actually a square.

### Second solution

The total perimeter length  $P$  of the original two rectangles is equal to the perimeter length of the large rectangle added to the lengths of the two edges which are joined together.



But the perimeter length of the large rectangle is  $\frac{2}{3}P$  and hence the two edges which are joined together have total length  $\frac{1}{3}P$ .

However, the two edges which are joined together are parallel to two sides of the large rectangle and have the same length as them. Hence these two sides of the large rectangle have total length  $\frac{1}{3}P$ .

Since the perimeter length of the large rectangle is  $\frac{2}{3}P$ , the other two sides of the large rectangle also have total length  $\frac{1}{3}P$ . It follows that all the sides of the rectangle are equal in length, in other words, the rectangle is a square.